

A SZEMERÉDI-TROTTER TYPE THEOREM IN \mathbb{R}^4

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ABSTRACT. We show that under suitable non-degeneracy conditions, m points and n 2-dimensional algebraic surfaces in \mathbb{R}^4 satisfying certain “pseudoflat” requirements can have at most $O(m^{2/3}n^{2/3} + m + n)$ incidences, provided that $m \leq n^{2-\epsilon}$ for any $\epsilon > 0$ (where the implicit constant in the above bound depends on ϵ), or $m \geq n^2$. As a special case, we obtain the Szemerédi-Trotter theorem for 2-planes in \mathbb{R}^4 , again provided $m \leq n^{2-\epsilon}$ or $m \geq n^2$. As a further special case we recover the Szemerédi-Trotter theorem for complex lines in \mathbb{C}^2 with no restrictions on m and n (this theorem was originally proved by Tóth using a different method). As a second special case, we obtain the Szemerédi-Trotter theorem for complex unit circles in \mathbb{C}^2 , which has applications to the complex unit distance problem. We obtain our results by combining the discrete polynomial ham sandwich theorem with the crossing number inequality.

1. INTRODUCTION

In [27], Szemerédi and Trotter proved the following theorem:

Theorem 1 (Szemerédi-Trotter). *The number of incidences between m points and n lines in \mathbb{R}^2 is $O(m^{2/3}n^{2/3} + m + n)$.*

Theorem 1 has seen a number of generalizations, including the following one due to Pach and Sharir in [21]:

Theorem 2 (Pach-Sharir). *Let \mathcal{P} be a collection of m points in \mathbb{R}^2 and \mathcal{S} a collection of n curves with k degrees of freedom—i.e. there exists a constant C_0 such that any two curves can meet in at most C_0 points and at most C_0 curves can contain any given k points. Then the number of incidences between points in \mathcal{P} and curves in \mathcal{S} is*

$$O(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + n),$$

where the implicit constant depends only on C_0 .

Theorem 2 was proved using the crossing number inequality for planar graphs, which was first employed in incidence geometry by Székely in [26] and later used by Arnov in [2] to obtain some partial incidence results in higher dimensions.

In a different direction, Tóth [29] generalized Theorem 1 to complex points and lines in \mathbb{C}^2 . However, as of this writing (2012), Tóth’s paper is still in the

midst of a lengthy review process while awaiting publication. Solymosi and Tardos [24] gave a simpler proof of the same result in the special case where the point set is a Cartesian product of the form $A \times B \subset \mathbb{C}^2$. Edelsbrunner and Sharir [8] obtained incidence results for certain configurations of points and codimension-1 hyperplanes in \mathbb{R}^4 , and Laba and Solymosi [17] obtained incidence bounds for points and a general class of 2-dimensional surfaces in \mathbb{R}^3 , provided the points satisfied a certain homogeneity condition. Elekes and Tóth [9] and later Solymosi and Tóth [25] obtained incidence results between points and hyperplanes in \mathbb{R}^d , again provided the points satisfied various nondegeneracy and homogeneity conditions.

In [23], Solymosi and Tao used the discrete polynomial ham sandwich theorem to obtain bounds for the number of incidences between points and bounded degree algebraic surfaces satisfying certain “pseudoflat” conditions (i.e. they behaved similarly to hyperplanes). Aside from an ϵ loss in the exponent, Solymosi and Tao’s result resolved a conjecture of Tóth on the number of incidences between points and d -flats in \mathbb{R}^{2d} . The discrete polynomial ham sandwich theorem was also used by the author in [31] to obtain incidence results between points and 2-dimensional surfaces in \mathbb{R}^3 (with no homogeneity condition), and by Kaplan et al. in [14] to obtain similar bounds on the number of incidences between points and spheres in \mathbb{R}^3 .

In this paper, we combine the crossing number inequality and the discrete polynomial ham sandwich theorem to obtain a new result which can be seen either as a sharpening of the \mathbb{R}^4 version of the Solymosi-Tao result from [23] or a generalization of Tóth’s result from [29]. To the best of the author’s knowledge, this is the first time these two techniques have been used together.

1.1. New Results.

Definition 3. We call a collection \mathcal{S} of surfaces C_0 -good if for each pair of distinct surfaces $S, S' \in \mathcal{S}$, we have the following:

- (i) $\dim S = 2$.
- (ii) S is a real algebraic variety and $\deg S \leq C_0$.
- (iii) $|S \cap S'| \leq C_0$.

Here, C_0 should be thought of as a fixed constant, while our collections of points and surfaces is allowed to be large. All of the bounds in the rest of this paper will depend on C_0 , but for notational convenience we shall suppress this dependence, and we shall say a collection of surfaces is *good* if it is C_0 -good for some fixed value of C_0 .

Definition 4. Let \mathcal{P} be a collection of points and \mathcal{S} a good collection of surfaces. We define the set of incidences of \mathcal{P} and \mathcal{S} to be

$$I(\mathcal{P}, \mathcal{S}) = \{(p, S) \in \mathcal{P} \times \mathcal{S} : p \in S\}.$$

Definition 5. Let $S \in \mathcal{S}$. We define S_{smooth} to be the set of smooth points of S (see e.g. [5]). If $x \in S_{\text{smooth}}$, we define $T_x(S)$ to be the tangent plane to S at x . This is an (affine) 2-plane in \mathbb{R}^4 .

Definition 6. We say a collection of incidences $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ is (C_0, k) -admissible if the following requirements are satisfied:

- (i) If $(p, S) \in \mathcal{I}$, then $p \in S_{\text{smooth}}$.
- (ii) If S and S' are distinct and $p \in S \cap S'$, then $T_p(S) \cap T_p(S') = p$.
- (iii) For any k points $p_1, \dots, p_k \in \mathcal{P}$, there are at most C_0 surfaces $S \in \mathcal{S}$ such that $(p_i, S) \in \mathcal{I}$ for each $i = 1, \dots, k$.

As above, we shall call a collection of incidences k -admissible if it is (C_0, k) -admissible for some value of C_0 . Finally, we may say a collection of surfaces is *admissible* if the value of k is either understood from the context or does not matter.

Example 7. The most important example arises from complex points and lines in \mathbb{C}^2 . Let $\tilde{\mathcal{P}}$ be a collection of (complex) points in \mathbb{C}^2 and let \mathcal{L} be a collection of complex lines in \mathbb{C}^2 . Then if $\mathcal{P} \subset \mathbb{R}^4$ is the image of $\tilde{\mathcal{P}}$ under the standard identification of \mathbb{C}^2 with \mathbb{R}^4 , and \mathcal{S} is the collection of affine 2-planes in \mathbb{R}^4 which are the images of the complex lines in \mathcal{L} , then \mathcal{S} is a 1-good collection of surfaces, and $I(\mathcal{P}, \mathcal{S})$ is a $(1, 2)$ -admissible collection of incidences.

Remark 8. Requirement (ii) in Definition 6 may seem artificial, but it is necessary to prevent the situation in which all of the surfaces meet in a common 1-dimensional variety and all of the points lie on that variety. If this were to occur, then the trivial bound of mn incidences would be sharp.

We are now ready to state our results.

Theorem 9. Let $\mathcal{P} \subset \mathbb{R}^4$ be a collection of points, with $|\mathcal{P}| = m$. Let \mathcal{S} be a good collection of surfaces, with $|\mathcal{S}| = n$. Let $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ be a collection of 2-admissible incidences. Then we have the following bounds.

- If $m \leq n^{2-\rho}$, then

$$|\mathcal{I}| \leq C_\rho (m^{2/3} n^{2/3} + m + n), \quad (1)$$

while if $m > n^2$,

$$|\mathcal{I}| \leq C_1 (m^{2/3} n^{2/3} + m + n). \quad (2)$$

- For all values of m and n ,

$$|\mathcal{I}| \leq C_1 (m^{2/3} n^{2/3} \log^{C_1} m + m + n). \quad (3)$$

Here, the constant C_1 may depend only on the constants appearing in Definitions 3 and 6, while C_ρ may also depend on ρ .

The next theorem is an analogue of Theorem 2, though we lose an additional logarithmic factor.

Theorem 9'. *Let $\mathcal{P} \subset \mathbb{R}^4$ be a collection of points, with $|\mathcal{P}| = m$. Let \mathcal{S} be a good collection of surfaces, with $|\mathcal{S}| = n$. Let $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ be a collection of k -admissible incidences. Then we have the following bounds.*

- If $m \leq n^{2-\rho}$, then

$$|\mathcal{I}| \leq C_\rho \left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n \right), \quad (1')$$

while if $m > n^2$,

$$|\mathcal{I}| \leq C_1 \left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + n \right). \quad (2')$$

- For all values of m and n ,

$$|\mathcal{I}| \leq C_1 \left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log^{C_1} m + m + n \right). \quad (3')$$

The constants C_1 and C_ρ have the same dependencies as in Theorem 9.

1.2. Corollaries and applications of Theorem 9. If \mathcal{S} is a collection of affine 2-planes in \mathbb{R}^4 with no two 2-planes meeting in a common line, then \mathcal{S} is a good collection. If $\mathcal{P} \subset \mathbb{R}^4$, then any collection of incidences $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ is 2-admissible. Thus we have the following corollary:

Corollary 10 (Szemerédi-Trotter for 2-planes in \mathbb{R}^4). Let $\mathcal{P} \subset \mathbb{R}^4$ be a collection of m points and let \mathcal{S} be a collection of n 2-planes in \mathbb{R}^4 such that any two planes meet in at most one point. Then the number of incidences between points in \mathcal{P} and planes in \mathcal{S} is bounded by the quantities in (2)–(3).

We can use Corollary 10 to recover the Szemerédi-Trotter theorem for complex lines in \mathbb{C}^2 , which was originally proved by Tóth in [29]. Note that by point-line duality in \mathbb{C}^2 , we can always assume that the number of lines is at least as great as the number of points. Thus we have:

Corollary 11 (Complex Szemerédi-Trotter). Let \mathcal{P} be a collection of m points and let \mathcal{S} be a collection of n (complex) lines in \mathbb{C}^2 . Then the number of incidences between points in \mathcal{P} and complex lines in \mathcal{S} is

$$O(m^{2/3}n^{2/3} + m + n).$$

As another special case, any collection \mathcal{S} of complex unit circles in \mathbb{C}^2 (regarded as 2-dimensional real varieties in \mathbb{R}^4) is a good collection. Furthermore, if \mathcal{P} is a collection of points then we can partition the collection of incidences between \mathcal{S} and \mathcal{P} into boundedly many classes $\mathcal{I}_1, \dots, \mathcal{I}_C$, such that each collection of incidences is 2-admissible. See Corollary 2.7 of [23] for details. Thus we have

Corollary 12 (Szemerédi-Trotter for complex unit circles). Let \mathcal{P} be a collection of m points and let \mathcal{S} be a collection of n (complex) unit circles in \mathbb{C}^2 . Then the number of incidences between points in \mathcal{P} and circles in \mathcal{S} is

$$O(m^{2/3}n^{2/3} + m + n).$$

This result allows us to obtain an improved bound on the unit distance problem in \mathbb{C}^2 :

Corollary 13 (Complex unit distance problem). Let $\mathcal{P} \subset \mathbb{C}^4$ be a finite set of points, with $|\mathcal{P}| = m$. Then

$$|\{(p, p') \in \mathcal{P}^2 : |p - p'| = 1\}| \lesssim m^{4/3}.$$

1.3. Proof sketch. For simplicity, we shall only sketch the proof of Theorem 9, i.e. the $k = 2$ case, and we will concentrate on proving (1). Furthermore, we shall assume that the surfaces in \mathcal{S} are 2-planes. The basic idea is as follows. Since each pair of points has at most two 2-planes passing through both of them, we can use the Cauchy-Schwarz inequality to obtain a rudimentary bound on the cardinality of any collection of point-surface incidences. We will call this the Cauchy-Schwarz bound. Using the discrete polynomial ham sandwich theorem, we find a polynomial P of controlled degree whose zero set cuts \mathbb{R}^4 into open “cells,” such that each cell contains roughly the same number of points from \mathcal{P} , and each surface from \mathcal{S} does not enter too many cells. We can then apply the Cauchy-Schwarz bound within each cell. This allows us to count the incidences occurring between surfaces and points in $\mathcal{P} \setminus Z$. In order to count the remaining incidences, we perform a “second level” polynomial ham sandwich decomposition on the variety Z . This gives us a polynomial Q which cuts Z into a collection of 3-dimensional cells, which are open in the relative (Euclidean) topology of Z . We then apply the Cauchy-Schwarz bound to each of these 3-dimensional cells. The only incidences left to count are those between surfaces in \mathcal{S} and points in $\mathcal{P} \cap Z \cap \{Q = 0\}$. Let $Y = Z \cap \{Q = 0\}$.

We can choose P and Q in such a way that Y is a 2-dimensional variety in \mathbb{R}^4 . Let S be a surface in \mathcal{S} . Then S will intersect Y in a union of isolated points (proper intersections) and 1-dimensional curves (non-proper intersections) (the case where S meets Y in a 2-dimensional variety can be dealt with easily). The number of isolated points in the intersection can be controlled by the degrees of the polynomials P and Q (we are working over \mathbb{R} , where Bézout’s theorem need not hold, so we need to be a bit careful), and thus the number of incidences between points $p \in \mathcal{P} \cap Y$ and surfaces $S \in \mathcal{S}$ such that p is an isolated point of $S \cap Y$ can be controlled.

The only remaining task is to control the number of incidences between points of $\mathcal{P} \cap Y$ and 1-dimensional curves arising from the intersection of Y and surfaces $S \in \mathcal{S}$. To simplify the exposition, pretend Y is a disjoint union of N 2-planes, i.e. $Y = \Pi_1 \sqcup \dots \sqcup \Pi_N$ (of course, we will not make this assumption when we prove the actual result). Then for each plane Π_i , $\Pi_i \cap S = L_{S,i}$ is a line on Π_i . It remains to count the number of incidences between $\mathcal{P} \cap \Pi_i$ and $\{L_{S,i}\}_{S \in \mathcal{S}}$. The Szemerédi-Trotter theorem for lines in \mathbb{R}^2 would give us the bound

$$I(\mathcal{P} \cap \Pi_i, \{L_{S,i}\}_{S \in \mathcal{S}}) \leq C|\mathcal{P} \cap \Pi_i|^{2/3}|\mathcal{S}|^{2/3} + |\mathcal{P} \cap \Pi_i| + |\mathcal{S}|, \quad (4)$$

but if we sum (4) over the N values of i , we have only bounded the number of incidences by $N^{1/3}|\mathcal{P}|^{2/3}|\mathcal{S}|^{2/3} + |\mathcal{P}| + |\mathcal{S}|$. Since N can be quite large, this is not sufficient. Instead, recall Székely’s proof in [26] of the Szemerédi-Trotter

theorem, which uses the crossing number inequality (the crossing number inequality and all other graph-related quantities are defined in Section 2.2 below). Loosely speaking, we consider the graph drawing G_i on Π_i whose vertices are the points of $\mathcal{P} \cap \Pi_i$, and two vertices are connected by an edge if there is a line segment from $\{L_{i,S}\}_{S \in \mathcal{S}}$ passing through the two points. Then the number of edges of the graph is comparable to the number of incidences between points and lines, and this is controlled by $\mathcal{C}(G_i)^{1/3} \mathcal{V}(G_i)^{2/3}$, where $\mathcal{C}(G_i)$ is the number of edge crossings and $\mathcal{V}(G_i)$ is the number of vertices of G_i . Thus in place of (4), we have

$$I(\mathcal{P} \cap \Pi_i, \{L_{S,i}\}_{S \in \mathcal{S}}) \leq C |\mathcal{P} \cap \Pi_i|^{2/3} |\mathcal{C}(G_i)|^{1/3} + |\mathcal{P} \cap \Pi_i| + |\mathcal{S}|. \quad (5)$$

The key insight is that

$$\sum_i |\mathcal{C}(G_i)| \leq |\mathcal{S}|^2. \quad (6)$$

Indeed, every pair of 2-planes $S, S' \in \mathcal{S}$ can intersect in at most one point, and since we assumed the planes $\{\Pi_i\}$ composing Y were disjoint, the intersection point of $S \cap S'$ can occur on Π_i for at most one index i . Summing (5) over all choices of i , applying Hölder's inequality, and inserting (6), we obtain the correct bound on the number of incidences between surfaces in \mathcal{S} and points lying on Y .

Unfortunately, the assumption that Y is a disjoint union of 2-planes need not be true. Thus we must cut Y up into pieces, each of which behaves like a 2-plane, and we need to prove a more general form of the Szemerédi-Trotter theorem for families of curves and points that lie on suitable domains. This is a purely topological argument, and it is done using the crossing number inequality. The decomposition of Y into suitable domains relies on results from real algebraic geometry, and it works well provided Y is not of too high degree. The degree of Y depends on the value of $(\log |\mathcal{P}|)/(\log |\mathcal{S}|)$. If $|\mathcal{P}|$ is too large compared to $|\mathcal{S}|$, the degree of Y is too big, and we cannot cut Y into suitable pieces without introducing error terms that are larger than the bounds we are trying to prove. Thus, instead of choosing “optimal” values for the degrees of P and Q , we choose smaller values so that the degree of Y is not too big. We can no longer apply the Cauchy-Schwarz bound to control the incidences occurring within each of the 3 and 4-dimensional cells described above. Instead, we borrow an idea from Solymosi and Tao in [23], and we assume an appropriate bound on the number of incidences between collections of points \mathcal{P}' and surfaces \mathcal{S}' , provided $|\mathcal{P}'|$ and $|\mathcal{S}'|$ have a better ratio of sizes than $|\mathcal{P}|$ and $|\mathcal{S}|$. Based on the way the cell decomposition works, this induction assumption will be satisfied for the collection of points and surfaces inside each cell of the decomposition, provided that $|\mathcal{P}| \leq |\mathcal{S}|^{2-\epsilon}$ for some constant $\epsilon > 0$, where the improvement obtained by the induction step depends on ϵ . After finitely many applications of the induction step, we obtain a situation where the ratio of $|\mathcal{P}|$ and $|\mathcal{S}|$ is very favorable, and thus our decomposition of Y into “2-plane-like pieces” is very good. At this point, we can select P and Q to have optimal degrees and use the

arguments above where we apply the Cauchy-Schwarz bound within each cell rather than the induction hypothesis.

The proof of (3) is similar, except we never apply the Cauchy-Schwarz bound; we keep applying the induction hypothesis bounds until our cells are so small that we can use a trivial bound to control the number of incidences that occur with each cell. This is similar to the techniques used by Solymosi and Tao. However, Solymosi and Tao need to iterate their induction step roughly $\log |\mathcal{P}|$ times, while with our techniques we only need to perform the induction step about $\log \log |\mathcal{P}|$ times. This allows us to get a slightly sharper bound. In order to prove Theorem 9', we use the more intricate crossing-number based arguments of Pach and Sharir from [21] in place of Székely's proof of the Szemerédi-Trotter theorem.

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2. PRELIMINARIES

In this section we shall recall some basic definitions and also state several lemmas that will be needed in the body of the proof. The proofs of these lemmas have been deferred to Appendix A.

2.1. Notation and Definitions. Throughout the paper, C will denote a sufficiently large absolute constant that is allowed to vary from line to line. We will write $A \lesssim B$ to mean $A < CB$ and we say that a quantity B is $O(A)$ if $B \lesssim A$. We say a quantity B is $o(A)$ if A is $O(B)$, and we say a quantity B is $o(1; \epsilon)$ if $B = B(\epsilon)$ is some function of ϵ , $\epsilon \geq 0$ with $B(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ (B may also be a function of additional variables. In this case, we require that $B(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ for each fixed value of the other variables).

2.2. Graph theory. Let G be a multigraph with maximum edge multiplicity M , and let H be a drawing of G , i.e. a collection of points and curves in \mathbb{R}^2 such that every vertex of G corresponds to a distinct point of H , and every edge of G corresponds to a curve in H such that every two curves intersect in a discrete set, and no points (i.e. vertices) are contained in the relative interior of any curve (i.e. edge).

Definition 14. We define $\mathcal{V}(G)$ to be the number of vertices of G and $\mathcal{E}(G)$ to be the number of edges, and similarly for H . We define $\mathcal{C}(H)$ to be the *crossing number* of H , i.e. the number of times two curves cross each other. Since the intersection of any two curves is a discrete set, $\mathcal{C}(H)$ is finite.

Theorem 15 (Ajtai, Chvatal, Newborn, Szemerédi [1]; Leighton [19]; Székely [26]). *Let H be a drawing of a multigraph with maximum edge multiplicity*

M. Then either $\mathcal{E}(H) < 5\mathcal{V}(H)$, or

$$\mathcal{C}(G) \geq \frac{\mathcal{E}(H)^3}{100M\mathcal{V}(H)^2}. \quad (7)$$

Theorem 16 (Kővari, Sós, Turan [16]). *Let s, t be fixed, and let $G = G_1 \sqcup G_2$ be a bipartite graph with $|G_1| = a, |G_2| = b$ that contains no copy of $K_{s,t}$. Then G has at most $O(ba^{1-1/s} + a)$ edges. Symmetrically, G has at most $O(ab^{1-1/t} + b)$ edges.*

2.3. Real algebraic geometry. Unless otherwise noted, all polynomials will be (affine) real polynomials, i.e. elements of $\mathbb{R}[x_1, \dots, x_d]$, and all varieties will be affine real varieties. Definitions and standard results about real algebraic varieties can be found in [4, 5].

Definition 17. A (real) algebraic variety $Z \subset \mathbb{R}^d$ is a set of the form $Z = \bigcap_{i=1}^{\ell} \{P_i = 0\}$, where $P_1, \dots, P_{\ell} \in \mathbb{R}[x_1, \dots, x_d]$ are polynomials. Note that we do not require varieties to be irreducible.

If $I \subset \mathbb{R}[x_1, \dots, x_d]$ is an ideal, we define

$$\mathbf{Z}(I) = \{x \in \mathbb{R}^d : f(x) = 0 \text{ for all } f \in I\}.$$

By abuse of notation, if $P \in \mathbb{R}[x_1, \dots, x_d]$, then we define $\mathbf{Z}(P) = \mathbf{Z}((P))$.

If $Z \subset \mathbb{R}^d$ is a variety, we define

$$\mathbf{I}(Z) = \{P \in \mathbb{R}[x_1, \dots, x_d] : P(x) = 0 \text{ for all } x \in Z\}.$$

Definition 18. If $Z \subset \mathbb{R}^d$ is a real variety, then $Z^* \subset \mathbb{C}^d$ denotes the smallest complex variety containing Z . Note that if $p \in \mathbb{R}^d$ is a point, then p^* is the image of p under the canonical embedding $\mathbb{R}^d \hookrightarrow \mathbb{C}^d$. Conversely, if $Z \subset \mathbb{C}^d$ is a complex variety, then $\Re(Z) \subset \mathbb{R}^d$ is its set of real points. The properties of Z^* and $\Re(Z)$ will be discussed further in Section 2.6.

Definition 19. Let $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_d]$ be a collection of non-zero real polynomials. A *strict sign condition* on \mathcal{Q} is a map $\sigma : \mathcal{Q} \rightarrow \{\pm 1\}$. If $Q \in \mathcal{Q}$, we will denote the evaluation of σ at Q by σ_Q . If $Z \subset \mathbb{R}^d$ is a variety, and σ is a strict sign condition on \mathcal{Q} , then we can define the *realization of σ on Z* by

$$\text{Real}(\sigma, \mathcal{Q}, Z) = \{x \in Z : Q(x)\sigma_Q > 0 \text{ for all } Q \in \mathcal{Q}\}. \quad (8)$$

We define

$$\Sigma_{\mathcal{Q}, Z} = \{\sigma : \text{Real}(\sigma, \mathcal{Q}, Z) \neq \emptyset\}. \quad (9)$$

and

$$\text{Real}(\mathcal{Q}, Z) = \{\text{Real}(\sigma, \mathcal{Q}, Z) : \sigma \in \Sigma_{\mathcal{Q}, Z}\}. \quad (10)$$

We call $\text{Real}(\mathcal{Q}, Z)$ the collection of “realizations of realizable strict sign conditions of \mathcal{Q} on Z .” Note that if some $Q \in \mathcal{Q}$ vanishes identically on Z then $\Sigma_{\mathcal{Q}, Z} = \emptyset$ and thus $\text{Real}(\mathcal{Q}, Z) = \emptyset$.

Definition 20. An ideal $I \subset \mathbb{R}[x_1, \dots, x_d]$ is *real* if for every sequence $a_1, \dots, a_{\ell} \in \mathbb{R}[x_1, \dots, x_d]$, $a_1^2 + \dots + a_{\ell}^2 \in I$ implies $a_j \in I$ for each $j = 1, \dots, \ell$.

The following proposition shows that real principal prime ideals and their corresponding real varieties have some of the “nice” properties of ideals and varieties defined over \mathbb{C} .

Proposition 21 (see [5, §4.5]). Let $(P) \subset \mathbb{R}[x_1, \dots, x_d]$ be a principal prime ideal. Then the following are equivalent:

- (i) (P) is real.
- (ii) $(P) = \mathbf{I}(\mathbf{Z}(P))$.
- (iii) $\dim(\mathbf{Z}(P)) = d - 1$.
- (iv) ∇P does not vanish identically on $\mathbf{Z}(P)$.
- (v) The sign of P changes somewhere on \mathbb{R}^d .

While not every polynomial $P \in \mathbb{R}[x_1, \dots, x_d]$ is a product of real ideals, the following lemma shows that for our applications, we can always modify our polynomials to ensure that this is the case.

Lemma 22. *Let $P \in \mathbb{R}[x_1, \dots, x_d]$ be a real polynomial. Then there exists a real polynomial \tilde{P} such that $\deg \tilde{P} \leq \deg P$, $\mathbf{Z}(P) \subset \mathbf{Z}(\tilde{P})$, and the irreducible components of \tilde{P} generate real ideals.*

The proof of this lemma has been deferred to Appendix A.

We will make essential use of the real Bézout’s theorem (see e.g. [4, §4.7]) and of Harnack’s theorem (see e.g. [7, p 57]).

Proposition 23 (Real Bézout). Let $P_1, \dots, P_d \in \mathbb{R}[x_1, \dots, x_d]$ be real polynomials of degrees D_1, \dots, D_d . Then the number of nonsingular intersection points of $\mathbf{Z}(P_1) \cap \dots \cap \mathbf{Z}(P_d)$ is at most $D_1 \dots D_d$.

Proposition 24 (Harnack). Let $Z \subset \mathbb{R}^2$ be an algebraic plane curve. Then Z contains $O((\deg Z)^2)$ connected components.

2.4. Polynomial ham sandwich-type theorems. In [11], Guth and Katz used the polynomial ham sandwich theorem to prove the following “discrete” polynomial ham sandwich theorem.

Theorem 25 (Discrete polynomial ham sandwich theorem). *Fix e . Let $F_1, \dots, F_\ell \subset \mathbb{R}^4$ be finite families of points, with $\ell \leq \binom{e}{4} - 1$. Then there exists a polynomial $P \in \mathbb{R}[x_1, \dots, x_4]$ of degree at most e such that*

$$|F_j \cap \{P > 0\}| \leq |F_j|/2 \quad \text{and} \quad |F_j \cap \{P < 0\}| \leq |F_j|/2, \quad j = 1, \dots, \ell.$$

Iterating Theorem 25 as in [11], for any discrete set we can find a polynomial P that cuts \mathbb{R}^4 into open “cells,” with each cell not containing too many points from the discrete set:

Theorem 26 (Polynomial ham sandwich cell decomposition). *Let \mathcal{P} be a collection of points in \mathbb{R}^4 , and let $D > 0$. Then there exists a non-zero polynomial P of degree at most D such that each connected component of $\mathbb{R}^4 \setminus \mathbf{Z}(P)$ contains at most $O(|\mathcal{P}|/D^4)$ points of \mathcal{P} .*

After applying Lemma 22, we can ensure that the irreducible components of P generate real ideals:

Corollary 27. Let \mathcal{P} be a collection of points in \mathbb{R}^4 , and let $D > 0$. Then there exists a non-zero polynomial P of degree at most D such that each connected component of $\mathbb{R}^4 \setminus \mathbf{Z}(P)$ contains at most $O(|\mathcal{P}|/D^4)$ points of \mathcal{P} , and each irreducible component of P generates a real ideal.

Example 28. Consider the following collection of 72 points:

$$\mathcal{P} = \bigcup_{j=1}^3 \{(\pm j, \pm j, \pm j, \pm j)\} \cup \bigcup_{j=1}^3 \{(0, \pm j, \pm j, \pm j)\}, \quad (11)$$

and let $D = 2$. Then the degree-4 polynomial

$$P(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$$

cuts \mathbb{R}^4 into 16 open cells $\Omega_1, \dots, \Omega_{16}$, (the cells are unbounded, but this is fine) plus the set

$$\mathbf{Z}(P) = \bigcup_{i=1}^4 \{x_i = 0\}.$$

We can verify that the polynomials x_1, \dots, x_4 generate real ideals, so P is a product of irreducible polynomials, each of which generates a real ideal. We have $|\Omega_i \cap \mathcal{P}| = 3 \leq |\mathcal{P}|/D^4$ for each $i = 1, \dots, 16$. Thus P satisfies the requirements of Corollary 27 (Corollary 27 only specifies the degree of P up to an implicit constant, so we cannot verify that the degree is correct). Finally, note that we have $|Z \cap \mathcal{P}| = 16$.

Example 29. Let $\mathcal{P} \subset \mathbb{R}^4$ be a large collection of points that lie in general position on the 2-plane $\{x_1 = x_2 = 0\}$, and let D be much smaller than $|\mathcal{P}|^{1/4}$. Then we can verify that the polynomial $P(x_1, x_2, x_3, x_4) = x_1$ satisfies the requirements of Corollary 27; $Z = \mathbf{Z}(P)$ cuts \mathbb{R}^4 into the two cells $\Omega_1 = \{x_1 > 0\}$ and $\Omega_2 = \{x_1 < 0\}$. We have $\Omega_1 \cap \mathcal{P} = \emptyset$ and $\Omega_2 \cap \mathcal{P} = \emptyset$. This phenomenon is unavoidable: any polynomial P satisfying the requirements of Corollary 27 must contain a factor P_1 of the form $P_1(x_1, x_2, x_3, x_4) = ax_1 + bx_2$ for some $a, b \in \mathbb{R}$ (provided the points of \mathcal{P} are in general position). Thus we must have $\mathcal{P} \subset \mathbf{Z}(P)$, so each of the cells of the decomposition $\mathbb{R} \setminus \mathbf{Z}(P)$ will be empty. This example is interesting because if $(\tilde{\mathcal{P}}, \mathcal{L})$ is a collection of complex points and lines in \mathbb{C}^2 , and $(\mathcal{P}, \mathcal{S})$ is the collection of points and 2-planes in \mathbb{R}^4 associated to $(\tilde{\mathcal{P}}, \mathcal{L})$, then all of the points of \mathcal{P} will lie on a common 2-plane, so the situation will resemble this example.

Theorem 26 will be used to obtain the “first level” decomposition of the point set \mathcal{P} . However, as seen in the above examples, many points may lie on the “boundary” $\mathbf{Z}(P)$, and we will need to control the number of incidences between surfaces in \mathcal{S} and points on $\mathbf{Z}(P)$. To do this, we shall perform a second discrete polynomial ham sandwich decomposition on the algebraic set $\mathbf{Z}(P)$. This shall be done with the following theorem:

Theorem 30 (Polynomial ham sandwich cell decomposition on a hypersurface). *Let \mathcal{P} be a collection of points in \mathbb{R}^4 lying on the set $Z = \mathbf{Z}(P)$ for P*

an irreducible polynomial of degree D such that P generates a real ideal. Let $\rho > 0$ be a small constant, and let $E \geq \rho D$. Then there exists a collection of polynomials $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_4]$ with the following properties:

- (i) $|\mathcal{Q}| \leq \log_2(DE^3) + O(1)$.
- (ii) $\sum_{Q \in \mathcal{Q}} \deg Q \lesssim E$.
- (iii) None of the polynomials in \mathcal{Q} vanish identically on Z .
- (iv) The realization of each of the $O(DE^3)$ strict sign conditions of \mathcal{Q} on Z contains at most $O(\frac{|\mathcal{P}|}{DE^3})$ points of \mathcal{P} .
- (v) Each irreducible component of each polynomial $Q \in \mathcal{Q}$ generates a real ideal.

All implicit constants depend only on ρ and the dimension d .

Theorem 30 is proved in [31, §A.3]. We can continue Examples 28 and 29.

Example 28'. Let \mathcal{P} , D , P , and Z be as in Example 28 above. Then $P_1(x_1, x_2, x_3, x_4) = x_1$ is the only irreducible component of P whose zero-set contains points from \mathcal{P} . Let $Z_1 = \mathbf{Z}(P_1) = \{x_1 = 0\}$, and let $E = 2$. Let $\mathcal{Q} = \{x_2, x_3, x_4\}$. Then $\text{Real}(\mathcal{Q}, Z_1)$ consists of the 8 octants of \mathbb{R}^3 , where we identify \mathbb{R}^3 with the hyperplane $\{x_1 = 0\}$ in \mathbb{R}^4 . Each of the components of $\text{Real}(\mathcal{Q}, Z_1)$ contains 2 points from $\mathcal{P} \cap Z_1$, and

$$\mathcal{P} \cap Z_1 \cap \bigcup_{j=2}^4 \{x_j = 0\} = \emptyset,$$

i.e. every point of \mathcal{P} either lies in some cell of $\mathbb{R}^4 \setminus Z$ or some realization of a realizable strict sign condition of \mathcal{Q} on Z_1 .

Example 29'. Let \mathcal{P} , D , P , and Z be as in Example 29, and let E be much smaller than $|\mathcal{P}|^{1/3}$. Let $\mathcal{Q} = \{x_2\}$. Then $\text{Real}(\mathcal{Q}, Z)$ consists of the sets $\{x_1 = 0, x_2 > 0\}$ and $\{x_1 = 0, x_2 < 0\}$. Neither of these sets contain any points from $\mathcal{P} \cap Z$; indeed, $\mathcal{P} \subset \{x_1 = x_2 = 0\}$. Thus \mathcal{Q} satisfies the requirements of Theorem 30, but none of the points of \mathcal{P} lie in any cell of $\mathbb{R}^4 \setminus Z$ nor in any realization of a realizable strict sign condition of \mathcal{Q} on Z . Section 3.4 will be devoted to dealing with this type of situation.

The following lemma will be used to control the number of incidences that can occur between points and good surfaces that lie inside of an algebraic variety in \mathbb{R}^4 . A proof of the lemma can be found in Appendix 33.

Lemma 31. Let \mathcal{S} be a collection of good surfaces, let \mathcal{P} be a collection of points, let $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ be a collection of admissible incidences, and let $V \subset \mathbb{R}^4$ be a variety with $V \neq \mathbb{R}^4$. Suppose that for each $S \in \mathcal{S}$, we have $S \subset V$, and that for each $(p, S) \in \mathcal{I}$ we have $p \in V_{\text{smooth}}$. Then

$$|\mathcal{I}| \lesssim |\mathcal{P}|. \tag{12}$$

2.5. Thom-type lemmas. In order to use the polynomial ham sandwich cell decomposition described above, we need to control how many cells a given surface $S \in \mathcal{S}$ can enter. This is accomplished by the following theorem:

Lemma 32. *Let $P \in \mathbb{R}[x_1, \dots, x_4]$ be a real polynomial of degree D whose irreducible components generate real ideals, and let S be a 2-dimensional irreducible real variety of degree $\leq C$. Then S intersects at most $O(D^2)$ connected components of $\mathbb{R}^d \setminus \mathbf{Z}(P)$, where the implicit constant depends only on C .*

We will need similar control on the number of times a surface $S \in \mathcal{S}$ can enter a 3-dimensional cell on the variety $\mathbf{Z}(P)$.

Lemma 33. *Let $P \in \mathbb{R}[x_1, \dots, x_4]$ be an irreducible real polynomial of degree D that generates a real ideal, and let $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_4]$ be a collection of real polynomials, such $\sum_{Q \in \mathcal{Q}} \deg Q = E$, and for each $Q \in \mathcal{Q}$, the irreducible components of Q generate real ideals. Let S be a 2-dimensional surface of degree $\leq C$. Then S meets at most $O(DE)$ realizations of realizable strict sign conditions of \mathcal{Q} on $\mathbf{Z}(P)$.*

The proofs of Lemmas 32 and 33 can be found in Appendix A.

Remark 34. A similar result to Lemma 33 can be obtained from the recent work of Barone and Basu in [3].

2.6. Real and Complex Varieties.

Proposition 35 (see [30, §10]). Let $Z \subset \mathbb{R}^d$ be a real variety and let $(Z^*)_1, \dots, (Z^*)_\ell$ be the irreducible components of Z^* . Then $\Re((Z^*)_1), \dots, \Re((Z^*)_\ell)$ are the irreducible components of Z . Furthermore, for each $j = 1, \dots, \ell$, $\Re((Z^*)_j)^* = (Z^*)_j$, so in particular $\Re((Z^*)_j)$ is non-empty.

Corollary 36. If $P, Q \in \mathbb{R}[x_1, \dots, x_d]$ are irreducible and $(P), (Q)$ are real ideals such that $\dim(\mathbf{Z}(P) \cap \mathbf{Z}(Q)) = d - 2$, then $\mathbf{Z}(P)^* \cap \mathbf{Z}(Q)^*$ is a complete intersection.

3. MAIN RESULTS: PROOF OF THEOREM 9

The proofs of Theorems 9 and 9' are very similar, so wherever possible we shall deal with both theorems at the same time. Thus we will frequently divide our arguments into the cases $k = 2$ (corresponding to Theorem 9) and $k \geq 3$ (corresponding to Theorem 9').

Let $\mathcal{P}, \mathcal{S}, \mathcal{I}$, and k be as in the statements of Theorems 9 and 9'. First, it suffices to prove Theorems 9 and 9' in the special case where all the surfaces $S \in \mathcal{S}$ are irreducible. If the surfaces are not irreducible, then there exists a constant C_0 such that each $S \in \mathcal{S}$ can be written $S = S_1 \cup S_2 \cup \dots \cup S_C$, $C \leq C_0$, where each S_i is an irreducible 2-dimensional surface (technically, S may contain irreducible components that are 0 or 1-dimensional, but by Property (i) of a admissible intersection, no points may lie on such components, so

we may discard them). For each index $j = 1, \dots, C_0$, let $\mathcal{S}_j = \{S_j : S \in \mathcal{S}\}$ (where $S_j = \emptyset$ if S contains fewer than j irreducible components). We can verify that each collection \mathcal{S}_j of surfaces is good, and that $\mathcal{I} \cap I(\mathcal{P}, \mathcal{S}_j)$ is admissible. We have

$$\mathcal{I} \subset \bigcup_j \mathcal{I} \cap I(\mathcal{P}, \mathcal{S}_j), \quad (13)$$

and the surfaces in each collection \mathcal{S}_j are irreducible. Thus if we can establish Theorem 9 in the special case where all the surfaces are irreducible, we can apply the result to each term in (13) to recover the general result. Thus we shall assume that all surfaces in \mathcal{S} are irreducible.

First, from Lemma 16, we have

$$|\mathcal{I}| \lesssim mn^{1-1/k} + n, \quad (14)$$

$$|\mathcal{I}| \lesssim m^{1/2}n + m. \quad (15)$$

This immediately gives us (2) and (2)', and it also gives us (1), (1)', (3), and (3) if $n > cm^k$ or $m > cn^2$ for some small constant c to be chosen later. Thus we may assume

$$\begin{aligned} n &< cm^k, \\ m &< cn^2. \end{aligned} \quad (16)$$

3.1. Main induction step. Theorem 9 will follow from the following two theorems:

Theorem 37 (Base case). *Let \mathcal{S} be a good collection of surfaces, \mathcal{P} a collection of points, and \mathcal{I} a collection of k -admissible incidences. Suppose*

$$|\mathcal{P}| \leq |\mathcal{S}|.$$

Then

$$|\mathcal{I}| \lesssim |\mathcal{P}|^{2/3} |\mathcal{S}|^{2/3} + |\mathcal{P}| + |\mathcal{S}| \quad (17)$$

if $k = 2$, and

$$|\mathcal{I}| \lesssim |\mathcal{P}|^{\frac{k}{2k-1}} |\mathcal{S}|^{\frac{2k-2}{2k-1}} \log |\mathcal{P}| + |\mathcal{P}| + |\mathcal{S}| \quad (17')$$

if $k \geq 3$.

Theorem 38 (Induction step). *Let \mathcal{S} be a good collection of surfaces, \mathcal{P} a collection of points, and \mathcal{I} a collection of k -admissible incidences. Then there exists the following:*

(i) *A number A satisfying*

$$|\mathcal{P}|^{c_0} < A < |\mathcal{P}|^{1/3} |\mathcal{S}|^{-1/6}, \quad (18)$$

where c_0 is some small absolute constant.

(ii) *Numbers $\{A_i\}$ with each $A_i \geq 0$ and*

$$\sum A_i \leq 100A, \quad (19)$$

and numbers $\{m_i\}$ with each $m_i \geq 0$ and

$$\sum m_i \leq |\mathcal{P}|. \quad (20)$$

(iii) Collections $\{(\mathcal{P}_j, \mathcal{S}_j)\}$, with $\mathcal{P}_j \subset \mathcal{P}$, $\mathcal{S}_j \subset \mathcal{S}$, with

$$|\mathcal{P}_j| \lesssim |\mathcal{P}|/A^4, \quad (21)$$

$$\sum_j |\mathcal{S}_j| \lesssim A^2 |\mathcal{S}|. \quad (22)$$

(iv) Collections $\{(\mathcal{P}_{ij}, \mathcal{S}_{ij})\}$, with $\mathcal{P}_{ij} \subset \mathcal{P}$, $\mathcal{S}_{ij} \subset \mathcal{S}$ such that for each index i ,

$$\begin{aligned} |\mathcal{P}_{ij}| &\lesssim m_i/(A_i A^3), \\ \sum_j |\mathcal{S}_{ij}| &\lesssim A_i A |\mathcal{S}|. \end{aligned} \quad (23)$$

These collections have the property that

$$\mathcal{I} \subset \left(\bigcup_j \mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j) \right) \cup \left(\bigcup_{ij} \mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij}) \right) \cup \mathcal{I}^*, \quad (24)$$

where

$$|\mathcal{I}^*| \leq C \left(|\mathcal{P}|^{2/3} |\mathcal{S}|^{2/3} + |\mathcal{P}| + n \right) \quad (25)$$

if $k = 2$, and

$$|\mathcal{I}^*| \leq C \left(|\mathcal{P}|^{\frac{k}{2k-1}} |\mathcal{S}|^{\frac{2k-2}{2k-1}} \log |\mathcal{P}| + |\mathcal{P}| + |\mathcal{S}| \right) \quad (25')$$

if $k \geq 3$.

Proof of Theorem 9 using Theorems 37 and 38. We will first prove the bounds (1) and (1') by establishing the following induction step. There exists a small absolute constant c_0 with the following property. If there exists some $\rho_0 > 0$ and a constant C_{ρ_0} for which (1) and (1') hold for that value of ρ_0 , then there exists a constant C^* for which (1) and (1') hold for all values of $\rho \leq (1 - c_0)\rho_0$. If $\rho > 1$ then (1) and (1') hold by Theorem 37. Once this fact has been established, it suffices to note that for any $\rho > 0$, if $B > \log \rho / \log(1 - c_0)$, then we obtain the bounds (1) and (1') for ρ after at most B iterations of the above procedure. Thus it suffices to establish the induction step.

Suppose (1) or (1') holds for all $\rho > \rho_0$. Let $\rho > (1 - c_0)\rho_0$. Apply the decomposition from Theorem 38. We can assume that for each index j ,

$$|\mathcal{S}_j| > |\mathcal{S}|/(1000CA^2), \quad (26)$$

where C is the implicit constant from (22). If this is not the case for some j , simply add admissible surfaces to \mathcal{S}_j at random. Doing so will not affect \mathcal{I} , and it will increase $\sum_j |\mathcal{S}_j|$ by at most a factor of 2.

Now, for each index j , we have

$$\begin{aligned} |\mathcal{P}_j| &< C|\mathcal{P}|A^{-4}, \\ |\mathcal{S}_j| &> \frac{1}{C}|\mathcal{S}|A^{-2}. \end{aligned}$$

Now, $|\mathcal{P}| \leq |\mathcal{S}|^{2-\rho}$, i.e. $C|\mathcal{P}_j|A^4 \leq (C|\mathcal{S}_j|A^2)^{2-\rho}$, thus

$$\begin{aligned} |\mathcal{P}_j|A^4 &\leq (C|\mathcal{S}_j|A^2)^{2-\rho}, \\ |\mathcal{P}_j| &\leq C^{2-\rho}A^{-2\rho}|\mathcal{S}_j|^{2-\rho}, \\ &\leq |\mathcal{S}_j|^{2-(1+c_0)\rho}, \end{aligned} \tag{27}$$

where in the final line we used (18) and the assumption that $|\mathcal{S}|$ is sufficiently large to absorb the constant $C^{2-\rho}$ into the exponent (if $|\mathcal{S}|$ is not sufficiently small then the entire result is trivial).

Similarly, for each index i, j , we have

$$\begin{aligned} |\mathcal{P}_{ij}| &< Cm_i A_i^{-1} A^{-3} < Cm A_i^{-1} A^{-3}, \\ |\mathcal{S}_{ij}| &> \frac{1}{C}|\mathcal{S}|A_i^{-1} A^{-1}. \end{aligned}$$

Thus $C|\mathcal{P}_{ij}|A_i A^3 \leq (C|\mathcal{S}_{ij}|A_i A)^{2-\rho}$, so

$$\begin{aligned} |\mathcal{P}_{ij}|A_i A^3 &\leq (C|\mathcal{S}_{ij}|A_i A)^{2-\rho}, \\ |\mathcal{P}_{ij}| &\leq C^{2-\rho}(AA_i)^{-\rho}(A_i/A)|\mathcal{S}_{ij}|^{2-\rho} \\ &\leq C^{2-\rho}A^{-\rho}|\mathcal{S}_{ij}|^{2-\rho} \\ &\leq |\mathcal{S}_{ij}|^{2-(1+c_0)\rho}. \end{aligned} \tag{28}$$

Now, recall that we selected ρ so that $(1+c_0)\rho > \rho_0$. Then by our induction hypothesis,

$$\begin{aligned} |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| &\leq C_{\rho_0} \left(|\mathcal{P}_j|^{2/3} |\mathcal{S}_j|^{2/3} + |\mathcal{P}_j| + |\mathcal{S}_j| \right), \\ |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| &\leq C_{\rho_0} \left(|\mathcal{P}_{ij}|^{2/3} |\mathcal{S}_{ij}|^{2/3} + |\mathcal{P}_{ij}| + |\mathcal{S}_{ij}| \right) \end{aligned} \tag{29}$$

if $k = 2$, and

$$\begin{aligned} |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| &\leq C_{\rho_0} \left(|\mathcal{P}_j|^{\frac{k}{2k-1}} |\mathcal{S}_j|^{\frac{2k-2}{2k-1}} \log |\mathcal{P}_j| + |\mathcal{P}_j| + |\mathcal{S}_j| \right), \\ |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| &\leq C_{\rho_0} \left(|\mathcal{P}_{ij}|^{\frac{k}{2k-1}} |\mathcal{S}_{ij}|^{\frac{2k-2}{2k-1}} \log |\mathcal{P}_{ij}| + |\mathcal{P}_{ij}| + |\mathcal{S}_{ij}| \right) \end{aligned} \tag{29'}$$

if $k \geq 3$. Thus if $k = 2$, we obtain the following bound (for clarity, we shall write the computation for general k):

$$\begin{aligned}
& \sum_j |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| \\
& \leq C_{\rho_0} \sum_j \left(|\mathcal{P}_j|^{\frac{k}{2k-1}} |\mathcal{S}_j|^{\frac{2k-2}{2k-1}} + |\mathcal{P}_j| + |\mathcal{S}_j| \right) \\
& \leq C_{\rho_0} \left[\left(\sum_j |\mathcal{P}_j|^k \right)^{\frac{1}{2k-1}} \left(\sum_j |\mathcal{S}_j| \right)^{\frac{2k-2}{2k-1}} + m + CA^2 n \right] \quad (30) \\
& \leq C_{\rho_0} \left[C m^{\frac{k}{2k-1}} A^{-\frac{4k-4}{2k-1}} n^{\frac{2k-2}{2k-1}} A^{\frac{4k-4}{2k-1}} + m + CA^2 n \right] \\
& \leq C_{\rho_0} \cdot C \left[m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + A^2 n \right],
\end{aligned}$$

and if $k \geq 3$,

$$\sum_j |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| \leq C_{\rho_0} \cdot C \left[m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + A^2 n \right]. \quad (30')$$

Similarly, if $k = 2$,

$$\begin{aligned}
& \sum_{ij} |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| \\
& \leq C_{\rho_0} \sum_{ij} \left(|\mathcal{P}_{ij}|^{\frac{k}{2k-1}} |\mathcal{S}_{ij}|^{\frac{2k-2}{2k-1}} + \sum_{ij} |\mathcal{P}_{ij}| + \sum_{ij} |\mathcal{S}_{ij}| \right) \\
& \leq C_{\rho_0} \left[\sum_i \left(\sum_j |\mathcal{P}_{ij}|^k \right)^{\frac{1}{2k-1}} \left(\sum_j |\mathcal{S}_{ij}| \right)^{\frac{2k-2}{2k-1}} + m + CA^2 n \right] \\
& \leq C_{\rho_0} \left[C \sum_i m_i^{\frac{k}{2k-1}} (A_i A^3)^{-\frac{k-1}{2k-1}} n^{\frac{2k-2}{2k-1}} (A_i A)^{\frac{2k-2}{2k-1}} + m + CA^2 n \right] \quad (31) \\
& \leq C_{\rho_0} \left[C n^{\frac{2k-2}{2k-1}} \sum_i m_i^{\frac{k}{2k-1}} (A_i/A)^{\frac{k-1}{2k-1}} + m + CA^2 n \right] \\
& \leq C_{\rho_0} \left[C n^{\frac{2k-2}{2k-1}} m^{\frac{k}{2k-1}} \left(\sum_i (A_i/A) \right)^{\frac{k-1}{2k-1}} + m + CA^2 n \right] \\
& \leq C_{\rho_0} \cdot C \left[n^{\frac{2k-2}{2k-1}} m^{\frac{k}{2k-1}} + m + A^2 n \right].
\end{aligned}$$

and if $k \geq 3$,

$$\begin{aligned}
& \sum_{ij} |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| \\
& \leq C_{\rho_0} \cdot C \left[n^{\frac{2k-2}{2k-1}} m^{\frac{k}{2k-1}} \log m + m + A^2 n \right]. \quad (31')
\end{aligned}$$

Now, by (18), $A^2 n \leq C n^{\frac{2k-2}{2k-1}} m^{\frac{k}{2k-1}}$, and thus for $k = 2$ we have

$$\begin{aligned} |\mathcal{I}| &\leq \sum_j |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| + \sum_{ij} |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| + |\mathcal{I}^*| \\ &\leq C_{\rho_0} \cdot C(m^{2/3} n^{2/3} + m + n), \end{aligned} \quad (32)$$

while for $k \geq 3$ we have

$$|\mathcal{I}| \leq C_{\rho_0} \cdot C(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n). \quad (32')$$

Thus, if we set $C_\rho = C \cdot C_{\rho_0}$, we obtain (1) and (1') and for all values of $\rho' \geq \rho$.

We can apply similar reasoning to establish (3) and (3'). The key idea is that by iterating Theorem 38 about $\log \log m$ times, we reduce to the case of counting incidences between collections of points and curves of bounded size, and we can apply trivial bounds. On the other hand, each application of Theorem 38 introduces a multiplicative constant C_0 into our bounds, so our final bounds are sharp except for a $C_0^{\log \log m} = \log^C m$ factor. We will now make this intuition precise. The case $k = 2$ and $k \geq 3$ can be dealt with simultaneously. Suppose (3) holds for all collections $\tilde{\mathcal{P}}, \tilde{\mathcal{S}}, \tilde{\mathcal{I}}$ with $|\tilde{\mathcal{P}}| \leq m_0$. Apply Theorem 38 to the collection $\mathcal{P}, \mathcal{S}, \mathcal{I}$. Then

$$\begin{aligned} &\sum_j |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| \\ &\leq C_0 \sum_j \left(|\mathcal{P}_j|^{\frac{k}{2k-1}} |\mathcal{S}_j|^{\frac{2k-2}{2k-1}} \log^C |\mathcal{P}_j| + |\mathcal{P}_j| + |\mathcal{S}_j| \right) \\ &\leq C_0 \left[\left(\sum_j |\mathcal{P}_j|^k \log^{C_1(2k-1)} |\mathcal{P}_j| \right)^{\frac{1}{2k-1}} \left(\sum |\mathcal{S}_j| \right)^{\frac{2k-2}{2k-1}} + m + A^2 n \right] \quad (33) \\ &\leq C \cdot C_0 \left[\left(A^{4-4k} m^k (\log m - 4 \log A)^{C_1(2k-1)} \right)^{\frac{1}{2k-1}} \left(A^2 n \right)^{\frac{2k-2}{2k-1}} \right. \\ &\quad \left. + m + A^2 n \right] \\ &\leq C \cdot C_0 m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} (1 - 4c_0)^{C_1} \log m + Cm + CA^2 n. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \sum_{ij} |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| \\
& \leq C_0 \sum_{ij} \left(|\mathcal{P}_{ij}|^{\frac{k}{2k-1}} |\mathcal{S}_{ij}|^{\frac{2k-2}{2k-1}} \log^{C_1} |\mathcal{P}_{ij}| + |\mathcal{S}_{ij}| + |\mathcal{P}_{ij}| \right) \\
& \leq C_0 \left(\sum_i \left(\sum_j |\mathcal{P}_{ij}|^k \log^{C_1(2k-1)} |\mathcal{P}_{ij}| \right)^{\frac{1}{2k-1}} \left(\sum_j |\mathcal{S}_{ij}| \right)^{\frac{2k-2}{2k-1}} + m + A^2 n \right) \\
& \leq C_0 \cdot C \sum_i \left(m_i^{\frac{k}{2k-1}} (A_i A^3)^{-\frac{k-1}{2k-1}} (\log(m_i/A_i A^3))^{C_1} (A_i A n)^{\frac{2k-2}{2k-1}} \right. \\
& \quad \left. + C(m + A^2 n) \right) \\
& \leq C_0 \cdot C n^{\frac{2k-2}{2k-1}} \sum_i m_i^{\frac{k}{2k-1}} (\log(m_i/A_i A^3))^{C_1} (A_i/A)^{\frac{k-1}{2k-1}} + C m + C A^2 n \\
& \leq C_0 \cdot C n^{\frac{2k-2}{2k-1}} \left(\sum_i m_i (\log(m_i/A_i A^3))^{C_1 \left(\frac{2k-1}{k} \right)} \right)^{\frac{k}{2k-1}} + C m + C A^2 n \\
& \leq C_0 \cdot C n^{\frac{2k-2}{2k-1}} m^{\frac{k}{2k-1}} (1 - 3c_0)^{C_1} \log^{C_1} m + C m + C A^2 n.
\end{aligned} \tag{34}$$

Let C_2 be the supremum of the constant appearing on the final lines of (33) and (34). If we make C_1 large enough,

$$C_2 \cdot (1 - 3c_0)^{C_1} < 1/3.$$

Using (18), and assuming m is sufficiently large (if m is not sufficiently large then the entire result is trivial), we have

$$C m + C A^2 n < \frac{1}{3} C_0 m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log^{C_1} m.$$

Thus,

$$\begin{aligned}
|\mathcal{I}| & \leq \sum_j |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_j)| + \sum_{ij} |\mathcal{I} \cap I(\mathcal{P}_{ij}, \mathcal{S}_{ij})| + |\mathcal{I}^*| \\
& \leq C_0 \left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log^{C_1} m + m + n \right).
\end{aligned} \tag{35}$$

This closes the induction. \square

3.2. The base case. In this section, we shall prove the base case for our induction.

Proof of Theorem 37. We shall prove the theorem by induction on m . For m sufficiently small the result is immediate, since we can chose a large constant in the quasi-inequalities (17) and (17').

3.2.1. *First ham sandwich decomposition.* Let

$$D = m^{\frac{k}{4k-2}} n^{\frac{-1}{2k-2}}, \quad (36)$$

which by (16) satisfies

$$C < D < n^{1/2}. \quad (37)$$

Let P be a squarefree polynomial of degree at most D such that $Z = \mathbf{Z}(P)$ cuts \mathbb{R}^4 into $O(D^4)$ cells $\{\Omega_i\}$, such that $|\mathcal{P} \cap \Omega_i| \lesssim m/D^4$. Let n_i be the number of surfaces in \mathcal{S} that meet cell i . By Lemma 32,

$$\sum n_i \lesssim D^2 n, \quad (38)$$

and thus if we apply Lemma 16 within each cell, we obtain

$$\begin{aligned} |\mathcal{I} \cap I(\mathcal{P} \setminus Z, \mathcal{S})| &\leq \sum_{i=1}^{O(D^4)} \mathcal{I}(\mathcal{P} \cap \Omega_i, \mathcal{S}) \\ &\lesssim \sum |\mathcal{P} \cap \Omega_i| n_i^{1-1/k} + n_i \\ &\lesssim \left(\sum |\mathcal{P} \cap \Omega_i|^k \right)^{1/k} \left(\sum n_i \right)^{1-1/k} + \sum n_i \\ &\lesssim \left(D^4 \frac{m^k}{D^{4k}} \right)^{1/k} \left(D^2 n \right)^{1-1/k} + D^2 n \\ &\lesssim \frac{mn^{1-1/k}}{D^{2-2/k}} + D^2 n. \end{aligned} \quad (39)$$

Write

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \quad (40)$$

where \mathcal{S}_1 (resp. \mathcal{S}_2) consists of those surfaces that are contained (resp. not contained) in Z . By Lemma 31, we have

$$|\mathcal{I} \cap I(\mathcal{P} \cap Z_{\text{smooth}}, \mathcal{S}_1)| \lesssim m, \quad (41)$$

so it suffices to consider incidences between surfaces and points lying on Z_{sing} . Let R be the square-free part of $|\nabla P|^2$, i.e. (R) is radical and $\mathbf{Z}(R) = \mathbf{Z}(|\nabla P|^2)$. Since P was square-free, $Z \cap \mathbf{Z}(R)$ is a complete intersection. If we let $\mathcal{S}'_1 \subset \mathcal{S}_1$ be those surfaces contained in $Z \cap \mathbf{Z}(R)$, then we must have $|\mathcal{S}'_1| \lesssim D^2$, and thus applying Lemma 16, we have

$$|\mathcal{I} \cap I(\mathcal{P} \cap Z_{\text{sing}}, \mathcal{S}'_1)| \lesssim D^2 m^{1/2} + m. \quad (42)$$

Let $\mathcal{S}'_2 \subset \mathcal{S}_1$ be those surfaces (contained in Z) that are not contained in $\mathbf{Z}(R)$. We must now control $\mathcal{I} \cap (\mathcal{P} \cap Z, \mathcal{S}_2)$ and $\mathcal{I} \cap (\mathcal{P} \cap \mathbf{Z}(R), \mathcal{S}'_2)$. But note that Z and $\mathbf{Z}(R)$ are both the zero-set of polynomials of degree $O(D)$, and thus the two collections of incidences can be dealt with in the same fashion.

3.2.2. *Second ham sandwich decomposition.* We shall now control $|\mathcal{I} \cap I(\mathcal{P} \cap Z, \mathcal{S}_2)|$. Factor $P = P_1 \dots, P_\ell$, with each P_j irreducible of degree D_j , and let $Z_j = \{P_j\}$. Let $\mathcal{P}_j \subset Z_j \cap \mathcal{P}$ so that $\bigsqcup \mathcal{P}_j = \mathcal{P}$; if the same point p lies on several Z_j , place p into just one of the sets \mathcal{P}_j (the choice of set does not matter). Let

$$\begin{aligned} \mathcal{A}_0 &= \{j : |\mathcal{P}_j|^k \leq cnD_j^{4k-2}\}, \\ \mathcal{A}_1 &= \{j : |\mathcal{P}_j|^2 > cnD_j^{4k-2}\}. \end{aligned}$$

We have

$$\begin{aligned} \left| \bigcup_{j \in \mathcal{A}_0} \mathcal{P}_j \right| &\leq c \sum_{j \in \mathcal{A}_0} n^{1/k} D_j^{4-2/k} \\ &\leq cm. \end{aligned} \tag{43}$$

Thus by the induction hypothesis, we have

$$|\mathcal{I} \cap I(\bigcup_{j \in \mathcal{A}_0} \mathcal{P}_j, \mathcal{S})| \leq C_0 (c^{\frac{k}{2k-1}} m^{2/3} n^{2/3} + m + n) \tag{44}$$

if $k = 2$, and

$$|\mathcal{I} \cap I(\bigcup_{j \in \mathcal{A}_0} \mathcal{P}_j, \mathcal{S})| \leq C_0 (c^{\frac{k}{2k-1}} m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n) \tag{44'}$$

if $k \geq 3$. If c is selected sufficiently small, the contribution from (44) is acceptable.

Let us now suppose $j \in \mathcal{A}_1$. Use Lemma 30 to find a collection \mathcal{Q}_j of polynomials with $\sum_{Q \in \mathcal{Q}_j} \deg Q \lesssim E_j$, where

$$E_j = |\mathcal{P}_j|^{\frac{k}{3k-2}} n^{\frac{-1}{3k-2}} D_j^{\frac{-k}{3k-2}}, \tag{45}$$

such that each strict sign condition of \mathcal{Q}_j on Z_j contains $O(|\mathcal{P}_j|/(D_j E_j^3))$ points.

Let m_{ij} be the number of points of \mathcal{P}_j that lie in the i -th strict sign condition of \mathcal{Q}_j on Z_j , and let n_{ij} be the number of surfaces in \mathcal{S}_2 that meet the i -th strict sign condition of \mathcal{Q}_j on Z_j . Let $W_j = \bigcup_{Q \in \mathcal{Q}_j} \mathbf{Z}(Q)$.

By Lemma 33,

$$\sum n_{ij} \lesssim n D_j E_j. \tag{46}$$

Thus

$$\begin{aligned}
& |\mathcal{I} \cap I(\mathcal{P}_j \setminus W_j, \mathcal{S}_2)| \\
& \lesssim \sum_i m_{ij} n_{ij}^{1-1/k} + \sum_i n_{ij} \\
& \lesssim \left(\sum_i m_{ij}^k \right)^{1/k} \left(\sum_i n_{ij} \right)^{1-1/k} + \sum_i n_{ij} \\
& \lesssim \left(D_j E_j^3 \frac{|\mathcal{P}_j|^k}{(D_j E_j^3)^k} \right)^{1/k} \left(n D_j E_j \right)^{1-1/k} + n D_j E_j \\
& \lesssim \frac{|\mathcal{P}_j| n^{1-1/k}}{E_j^{2-2/k}} + n D_j E_j.
\end{aligned} \tag{47}$$

It remains to control $|\mathcal{I} \cap I(\mathcal{P} \cap Z_j \cap W_j, \mathcal{S}_2)|$. We shall defer this step until the next section. Thus, if $j \in \mathcal{A}_1$ then

$$\begin{aligned}
|\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_2)| & \lesssim \frac{|\mathcal{P}_j| n^{1-1/k}}{E_j^{2-2/k}} + n D_j E_j + |\mathcal{P}_j| + |\mathcal{I} \cap I(\mathcal{P}, \mathcal{S}_2, Z_j \cap W_j)| \\
& \lesssim n^{\frac{3k-3}{3k-2}} |\mathcal{P}_j|^{\frac{k}{3k-2}} D_j^{\frac{2k-2}{3k-2}} + |\mathcal{P}_j| + |\mathcal{I} \cap I(\mathcal{P}, \mathcal{S}_2, Z_j \cap W_j)|,
\end{aligned} \tag{48}$$

and thus by Hölder's inequality,

$$\begin{aligned}
\sum_{j \in \mathcal{A}_1} |\mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}_2)| & \lesssim \sum_{j \in \mathcal{A}_1} n^{\frac{3k-3}{3k-2}} |\mathcal{P}_j|^{\frac{k}{3k-2}} D_j^{\frac{2k-2}{3k-2}} + \sum_{j \in \mathcal{A}_1} |\mathcal{P}_j| \\
& \quad + \sum_{j \in \mathcal{A}_1} |\mathcal{I} \cap I(\mathcal{P} \cap Z_j \cap W_j, \mathcal{S}_2)| \\
& \lesssim n^{\frac{3k-3}{3k-2}} m^{\frac{k}{3k-2}} D^{\frac{2k-2}{3k-2}} + m \\
& \quad + \sum_{j \in \mathcal{A}_1} |\mathcal{I} \cap I(\mathcal{P} \cap Z_j \cap W_j, \mathcal{S}_2)|.
\end{aligned} \tag{49}$$

Lemma 39. *Let Y be a 2-dimensional variety of the form:*

- $Y = \bigcup_j Y_j$, with $Y_j = Z_j \cap W_j$.
- $Z_j = \mathbf{Z}(P_j)$, $W_j = \mathbf{Z}(Q_j)$. $\deg P_j = D_j$, $\deg Q_j = E_j$.
- P_j is irreducible generates a real ideal. Q_j is a product of irreducible polynomials, each of generates a real ideal (and thus Z_j and Q_j are hypersurfaces).
- Q_j is not divisible by P_j .

Let $\mathcal{P} \subset Y$ be a collection of points, with $|\mathcal{P}| = m$. Let \mathcal{S} be a collection of admissible surfaces, with $|\mathcal{S}| = n$ such that no surface in \mathcal{S} is contained in Y . Let $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ be a collection of k -good incidences. Then the following holds:

- If $k = 2$, then

$$|\mathcal{I}| \lesssim m^{2/3} n^{2/3} + n \sum_j D_j E_j + \left(\sum_j D_j E_j \right)^4. \quad (50)$$

- If $k \geq 3$, then

$$|\mathcal{I}| \lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + n \log m \sum_j D_j E_j + \left(\sum_j D_j E_j \right)^4. \quad (51)$$

We shall defer the proof of Lemma 39 to Section 3.4 below. Applying Lemma 39 to the collection $(Y, \mathcal{P} \cap Y, \mathcal{S}_2)$, with $Y = \bigcup Z_j \cap W_j$, we see that if $k = 2$, then

$$\begin{aligned} \sum_{j \in \mathcal{A}_1} |\mathcal{I} \cap I(\mathcal{P} \cap Z_j \cap W_j, \mathcal{S}_2)| \\ \lesssim m^{2/3} n^{2/3} + n \sum_j D_j E_j + m + \left(\sum_j D_j E_j \right)^4 \\ \lesssim m^{2/3} n^{2/3} + m + n. \end{aligned} \quad (52)$$

Here we used the assumption $m \leq n$, and thus

$$\begin{aligned} \left(\sum_j D_j E_j \right)^4 &\lesssim \left(n^{-1/4} \sum_j m_j^{1/2} D_j^{1/2} \right)^4 \\ &\lesssim \left(n^{-1/4} m^{1/2} D^{1/2} \right)^4 \\ &\lesssim (m^2/n)^{4/3} \\ &\lesssim m^{2/3} n^{2/3}. \end{aligned}$$

If $k \geq 3$ then a similar computation shows

$$\sum_{j \in \mathcal{A}_1} |\mathcal{I} \cap I(\mathcal{P} \cap Z_j \cap W_j, \mathcal{S}_2)| \lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n. \quad (53)$$

Combining (44), (49), and (52) we obtain

$$|\mathcal{I} \cap I(\mathcal{P} \cap Z, \mathcal{S}_2)| \lesssim m^{2/3} n^{2/3} + m + n. \quad (54)$$

if $k = 2$, and

$$|\mathcal{I} \cap I(\mathcal{P} \cap Z, \mathcal{S}_2)| \lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n \quad (54')$$

if $k \geq 3$.

However, an identical argument can be used to obtain analogous bounds for $|\mathcal{I} \cap ((\mathcal{P} \cap \mathbf{Z}(R)), \mathcal{S}'_2)|$. Combining (16), (39), (41), (42), and either (54) or (54'), we obtain

$$\begin{aligned} |\mathcal{I}| &\lesssim m^{2/3} n^{2/3} + m + n + D^2 m^{1/2} \\ &\lesssim m^{2/3} n^{2/3} + m + n. \end{aligned} \quad (55)$$

if $k = 2$, and

$$|\mathcal{I}| \lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n. \quad (55')$$

if $k \geq 3$. \square

3.3. The induction step. We shall now prove the induction step theorem.

Proof of Theorem 38. The proof will follow a similar outline to that of Theorem 37, and we shall gloss over those steps that are identical.

3.3.1. First ham sandwich decomposition. Let

$$A = \left[m^{\frac{k}{4k-2}} n^{\frac{-1}{4k-2}} \right]^{1/2}. \quad (56)$$

By (16), A satisfies

$$\begin{aligned} C < A < n^{1/2}, \\ A &\leq m^{1/8}. \end{aligned} \quad (57)$$

As in the previous section, let P be a squarefree polynomial of degree at most A such that $Z = \mathbf{Z}(P)$ cuts \mathbb{R}^4 into $O(A^4)$ cells $\{\Omega_i\}$ such that if $\tilde{\mathcal{P}}_i = \Omega_i \cap \mathcal{P}$ then $|\tilde{\mathcal{P}}_i| \lesssim |\mathcal{P}|/A^4$. Let $\mathcal{S}_i = \{S \in \mathcal{S} : S \cap \Omega_i \neq \emptyset\}$. By Lemma 32, $\sum |\mathcal{S}_i| \lesssim A^2 n$. We have that the collections $\{\tilde{\mathcal{P}}_i\}, \{\mathcal{S}_i\}$ satisfy the requirements from (21) and (22).

It remains to find appropriate collections of points and surfaces that contain $\mathcal{I} \cap I(\mathcal{P} \cap Z, \mathcal{S})$. Write $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$ as in (40). Let

$$\mathcal{I}'_1 = |\mathcal{I} \cap I(\mathcal{P} \cap Z_{\text{smooth}}, \mathcal{S}_1)|.$$

By Lemma 31,

$$|\mathcal{I}'_1| \lesssim m, \quad (58)$$

We now consider incidences between surfaces and points lying on Z_{sing} . Let R be the squarefree part of $|\nabla P|^2$. Let $\mathcal{S}'_1 \subset \mathcal{S}_1$ be those surfaces contained in $Z \cap \mathbf{Z}(R)$, so $|\mathcal{S}'_1| \lesssim D^2$. Let

$$\mathcal{I}'_2 = |\mathcal{I} \cap I(\mathcal{P} \cap Z_{\text{sing}}, \mathcal{S}'_1)|.$$

By Lemma 16, we have

$$|\mathcal{I}'_2| \lesssim D^2 m^{1/2} + m. \quad (59)$$

Let $\mathcal{S}'_2 \subset \mathcal{S}_1$ be those surfaces (contained in Z) that are not contained in $\mathbf{Z}(R)$. We must now control $|\mathcal{I} \cap I(\mathcal{P} \cap Z, \mathcal{S}_2)|$ and $|\mathcal{I} \cap I(\mathcal{P} \cap \mathbf{Z}(R), \mathcal{S}'_2)|$.

3.3.2. Second ham sandwich decomposition. Factor $P = P_1 \dots, P_\ell$, and partition $\mathcal{P} = \mathcal{P}_1 \sqcup \dots \sqcup \mathcal{P}_j$ as in the previous section. Let $A_j = \deg P_j$.

For each index j , apply Lemma 30 to the data $(\mathbf{Z}(P_j), \mathcal{P}_j, E)$, and denote the resulting collection \mathcal{Q}_j .

Let \mathcal{P}_{ij} be those points of \mathcal{P}_j that lie in the i -th strict sign condition of \mathcal{Q}_j on Z_j , and let \mathcal{S}_{ij} be those surfaces in \mathcal{S}_2 that meet the i -th strict sign condition of \mathcal{Q}_j on Z_j . We can verify that $\{\mathcal{P}_{ij}\}$ and $\{\mathcal{S}_{ij}\}$ satisfy the requirements from (23).

We shall perform a similar decomposition for the points of $(\mathcal{P} \cap \mathbf{Z}(R)) \setminus Z$, and denote the resulting collections $\{\mathcal{Q}'_j\}, (\mathcal{P}'_{ij}, \mathcal{S}'_{ij})$. Again, $\{\mathcal{P}'_{ij}\}$ and $\{\mathcal{S}'_{ij}\}$ satisfy the requirements from (23), and thus the collection $\{(\mathcal{P}_{ij}, \mathcal{S}_{ij})\} \cup \{(\mathcal{P}'_{ij}, \mathcal{S}'_{ij})\}$ satisfies the properties from (23).

Finally, it remains to control the “boundary incidences,”

$$\begin{aligned}\mathcal{I}'_3 &= \mathcal{I} \cap I(\mathcal{P} \cap Y, \mathcal{S}), \\ \mathcal{I}'_4 &= \mathcal{I} \cap I(\mathcal{P} \cap Y', \mathcal{S}),\end{aligned}$$

where $Y = \bigcup_j Z_j \cap \bigcup_{Q \in \mathcal{Q}_j} \mathbf{Z}(Q)$, and Y' is defined similarly. But by Lemma 39,

$$\begin{aligned}|\mathcal{I}'_3| &\lesssim m^{2/3} n^{2/3} + n \sum_j D_j D + \left(\sum_j D_j D \right)^4 \\ &\lesssim m^{2/3} n^{2/3} + m + n,\end{aligned}\tag{60}$$

if $k = 2$, and

$$|\mathcal{I}'_3| \lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} \log m + m + n\tag{60'}$$

if $k \geq 3$. The analogous bounds hold for $|\mathcal{I}'_4|$. Thus if we let

$$\mathcal{I}' = \bigcup_{i=1}^4 \mathcal{I}'_i,$$

we can verify that our collections $\{\mathcal{P}_j\}$, $\{\mathcal{S}_j\}$, $\{\mathcal{P}_{ij}\}$, $\{\mathcal{S}_{ij}\}$, and \mathcal{I}' , satisfy the conclusions of Theorem 38. \square

3.4. Controlling boundary incidences: proof of Lemma 39. Before we begin the proof, let us first recall Székely’s proof in [26] of the Szemerédi-Trotter theorem, which uses the crossing number inequality. Let $\tilde{\mathcal{P}}$ be a collection of points and \mathcal{L} a collection of lines in \mathbb{R}^2 . Let $I(\tilde{\mathcal{P}}, \mathcal{L})$ be the number of incidences between points in $\tilde{\mathcal{P}}$ and lines in \mathcal{L} . Suppose that all of the points (and thus all of the incidences) are contained in some large disk $U \subset \mathbb{R}^2$. Consider the following graph drawing H : the vertices of H are the points of $\tilde{\mathcal{P}}$ and the points where a line from \mathcal{L} meets ∂U . The edges of H are the line segments connecting two vertices that arise from lines in \mathcal{L} . To each incidence $(p, L) \in I$, we can associate an edge of H in such a way that the same edge is assigned to at most two incidences. Thus $I(\tilde{\mathcal{P}}, \mathcal{L}) \lesssim \mathcal{E}(H)$. Now, delete all of the edges involving a vertex on ∂U , and delete the vertices

on ∂U . We have deleted at most $2|\mathcal{L}|$ edges. Let H' be the resulting graph drawing. Then $I(\tilde{\mathcal{P}}, \mathcal{L}) \lesssim \mathcal{E}(H') + 2|\mathcal{L}|$. By the crossing number inequality,

$$\begin{aligned} \mathcal{E}(H') &\lesssim \mathcal{V}(H') + \mathcal{C}(H')^{1/3} \mathcal{V}(H')^{2/3} \\ &\lesssim |\tilde{\mathcal{P}}| + |\mathcal{L}|^{2/3} |\tilde{\mathcal{P}}|^{2/3}. \end{aligned}$$

Thus $I(\tilde{\mathcal{P}}, \mathcal{L}) \lesssim |\tilde{\mathcal{P}}|^{2/3} |\mathcal{L}|^{2/3} + |\tilde{\mathcal{P}}| + |\mathcal{L}|$.

We wish to do the same thing on the real algebraic varieties $\{Y_j\}$. In order to do that, we will have to formulate the above proof in a slightly more general setting.

Definition 40. Let A be a collection of open curves (i.e. homeomorphic images of $(0, 1)$) and B a collection of points on a planar domain $U \subset \mathbb{R}^2$. Then we define the number of incidences between curves in A and points in B to be

$$I(A, B) = \{(\alpha, p) \in A \times B : p \in \bar{\alpha}\},$$

and we define the number of crossings of curves in B to be

$$\mathcal{C}(U) = |\{(\alpha, \alpha', p) \in A^2 \times U : p \in \alpha \cap \alpha'\}|.$$

Lemma 41 (Szemerédi-Trotter on a domain). *Let:*

- U be a smooth 2-dimensional manifold that is homeomorphic to an open subset of \mathbb{R}^2 .
- A be a collection of 1-dimensional open curves lying on U .
- B be a collection of points on U .
- Let C_0 be a constant so that for any two points $p, p' \in B$, there are at most C_0 curves $\alpha_1, \dots, \alpha_{C_0} \in A$ with $p \in \alpha_j$ and $p' \in \alpha_j$ for each $j = 1, \dots, C_0$.

Then

$$|I(A, B)| \lesssim |B|^{2/3} \mathcal{C}(U)^{1/3} + |B| + |A|, \quad (61)$$

where the implicit constant depends only on C_0 .

Remark 42. Note that in Definition 40, a point is incident to an (open) curve if it lies on the *closure* of that curve. On the other hand, a crossing of two curves is a point common to the relative interior of both curves. Similarly, in Lemma 41 we require that for any two points, there are at most $O(1)$ curves which contain those points in their relative interiors—there may be arbitrarily many curves whose closures contain the two points.

We also have the slightly weaker bound for “curves with k degrees of freedom”:

Lemma 41'. *Let U, A, B , and $\mathcal{C}(U)$ be as in Lemma 41, except in place of the final item we require: for any k points $p_1, \dots, p_k \in B$, there are at most C_0 curves from A that contain p_1, \dots, p_k . Then*

$$|I(A, B)| \lesssim |B|^{\frac{k}{2k-1}} \mathcal{C}(U)^{\frac{k-1}{2k-1}} + |B| \log |A| + |A|. \quad (61')$$

Lemmas 41 and 41' are useful when combined with the following lemma:

Lemma 43 ($\mathbb{R}\text{-Alg} \longrightarrow C^0$). *Let $Y \subset \mathbb{R}^2$ be a 2-dimensional real variety of the following form:*

- $Y = \bigcup_j Y_j$, where each $Y_j = Z_j \cap W_j$ is a 2-dimensional real algebraic set that is a (set-theoretic) complete intersection of two (real) hypersurfaces
- $Z_j = \mathbf{Z}(P_j)$, $W_j = \mathbf{Z}(Q_j)$, with $\deg P_j = D_j$, $\deg Q_j = E_j$, $E_j > cD_j$ for some small absolute constant c .

Let $\mathcal{P} \subset \mathbb{R}^4$ be a collection of points, let \mathcal{S} be a collection of good surfaces, and let \mathcal{I} be a collection of admissible incidences. Suppose further that for each $(p, S) \in \mathcal{I}$, there does not exist an index j for which p is an isolated point of $S \cap Y_j$. Thus for each $S \in \mathcal{S}$ and for index j , either $p \notin S \cap Y_j$, or p lies on a 1-dimensional component of $S \cap Y_j$.

Then there exists an “error set” $\mathcal{I}' \subset \mathcal{I}$ and an “incidence model” $\mathcal{M} = \{(U_i, A_i, B_i, \iota_i)\}_{i=1}^M$, where for each i ,

- $U_i \subset \mathbb{R}^4$ is homeomorphic to an open subset of \mathbb{R}^2 .
- A_i is a collection of open curves (homeomorphic to $(0, 1)$) contained in U_i .
- $B_i \subset U_i$ is a collection of points.
- $\iota_i: U_i \hookrightarrow \mathbb{R}^4$ is an embedding.

\mathcal{M} and \mathcal{I}' have the following properties:

(i) \mathcal{M} does not increase crossing number:

$$\sum_i \sum_{\alpha, \alpha' \in A_i} |\alpha \cap \alpha'| \leq \sum_{S, S' \in \mathcal{S}} |S \cap S'|. \quad (62)$$

(ii) \mathcal{M} counts incidences: If $(p, S) \in \mathcal{I} \setminus \mathcal{I}'$, then there exists some index i , some $\tilde{p} \in B_i$, and some $\alpha \in A_i$ such that $\iota_i(\tilde{p}) = p$, $p \in \iota_i(\alpha)$, $\alpha = \iota_i^{-1}(S)$, and $\tilde{p} \in \bar{\alpha}$.

(iii) The curves in A_i have k degrees of freedom: For each index i , and for each collection of k points $p_1, \dots, p_k \in B_i$, there are at most C_0 curves in A_i passing through each of p_1, \dots, p_k , where C_0 is the constant from Definition 3.

(iv) \mathcal{M} does not contain too many curves:

$$\sum_i |A_i| \lesssim |\mathcal{S}| \sum_j D_j E_j. \quad (63)$$

(v) \mathcal{I}' is not too big:

$$|\mathcal{I}'| \lesssim \left(\sum_j D_j E_j \right)^4 + m. \quad (64)$$

Using Lemmas 41 and 43, we shall prove Lemma 39.

Proof of Lemma 39.

Definition 44. Let $\{Y_j\}$, \mathcal{S} , \mathcal{P} , and \mathcal{I} be as in the statement of Lemma 39. We define

$$\begin{aligned} \mathcal{I}_0 = \{(p, S) \in \mathcal{I} : \text{there exists an index } j \text{ such that } p \\ \text{is an isolated point of } S \cap Y_j\}, \end{aligned} \quad (65)$$

$$\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0. \quad (66)$$

Note that since $S \not\subset Y_j$ for any $S \in \mathcal{S}$, \mathcal{I}_1 consists of those pairs (p, S) such that for each index j , either $p \notin S \cap Y_j$ or p lies on a 1-dimensional component of $S \cap Y_j$.

We shall first deal with \mathcal{I}_0 . Fix a choice of S and j . Since $S \not\subset Z_j$, $S \cap Z_j$ consists of 0 and 1-dimensional irreducible components. By Harnack's theorem (see Remark 45 below), $S \cap Z_j$ can contain at most D_j^2 0-dimensional components. Now, consider the collection Υ of 1-dimensional irreducible components of $S \cap Z_j$. We have $\sum_{\gamma \in \Upsilon} \deg \gamma \lesssim D_j$. For each curve $\gamma \in \Upsilon$, either $\gamma^* \subset W_j^*$, or $\gamma^* \cap W_j^*$ is a discrete set. Let Υ_1 and Υ_2 denote those sets where the former (resp. latter) occurs. If $\gamma \in \Upsilon_1$, then we must have $\gamma \subset Y_j$, and thus for any point $p \in \mathcal{P}$ that lies on γ , we must have $(p, S) \in \mathcal{I}_1$. If $\gamma \in \Upsilon_2$, then by Bézout's theorem (over \mathbb{C}), $|\gamma^* \cap W_j^*| \leq \deg \gamma \cdot E_j$, and thus there are at most $O(\deg \gamma \cdot E_j)$ points $p \in \mathcal{P}$ for which $p \in \gamma$ and $(p, S) \in \mathcal{I}_0$. Summing over all curves in Υ_2 , we conclude

$$\sum_{\gamma \in \Upsilon_2} \{(p, S) \in \mathcal{I}_0 : p \in \gamma\} \lesssim D_j E_j. \quad (67)$$

Remark 45. while Harnack's theorem deals with plane algebraic curves and $S \cap Z_j$ is a space curve, since $S \cap Z_j$ is a proper intersection, we can select a generic projection $\pi: \mathbb{R}^4 \rightarrow \mathbb{R}^2$. The image curve $\pi(S \cap Z_j)$ may have fewer connected components than the original curve, but the decrease is controlled by the number of singular points of $\pi(S \cap Z_j)$, and this in turn is controlled by D_j^2

Summing (67) over all indices j and all choices of $S \in \mathcal{S}$ and adding back the incidences that lie on 0-dimensional components of $S \cap Z_j$ for all choices of $S \in \mathcal{S}$, we obtain

$$|\mathcal{I}_0| \lesssim n \sum_j (D_j^2 + D_j E_j). \quad (68)$$

It remains to control $|\mathcal{I}_1|$. Use Lemma 43 on the data $(Y, \mathcal{P} \cap Y, \mathcal{S}, \mathcal{I}_1)$. Denote the resulting incidence model \mathcal{M} and the resulting error set \mathcal{I}' . We

have

$$\sum_i |I(A_i, B_i)| \geq |\mathcal{I}_1|, \quad (69)$$

$$\sum_i |B_i| \leq \sum_j |\mathcal{P}_j|, \quad (70)$$

$$\sum_i |A_i| \leq |\mathcal{S}| \sum_j D_j E_j, \quad (71)$$

$$\sum_i \mathcal{C}(U_i) \leq \sum_{S, S'} |S \cap S'|. \quad (72)$$

(69), (70), and (72) follow from the definition of \mathcal{M}_j and the observation that if $\alpha \in A_i$, $|\partial\alpha| = 2$, i.e. an open curve only has two ends.

Now, if $k = 2$, apply Lemma 41 to each collection (U_i, A_i, B_i) . We have

$$\begin{aligned} |\mathcal{I}_1| &\lesssim \sum_i |I(A_i, B_i)| + |\mathcal{I}'| \\ &\lesssim \sum_i |B_i|^{2/3} \mathcal{C}(U_i)^{1/3} + \sum_i |A_i| + \sum_i |B_i| + |\mathcal{I}'| \\ &\lesssim \left(\sum_i |B_i| \right)^{2/3} \left(\sum_i \mathcal{C}(U_i) \right)^{1/3} \\ &\quad + \sum_i |A_i| + \sum_i |B_i| + |\mathcal{I}'| \\ &\lesssim m^{2/3} n^{2/3} + n \sum_j D_j E_j + m + \left(\sum_j D_j E_j \right)^4, \end{aligned} \quad (73)$$

where in the final line we used (62) and the fact that

$$\sum_{S, S'} |S \cap S'| \lesssim n^2.$$

If $k \geq 3$, apply Lemma 41' to each collection (U_i, A_i, B_i) . We get

$$|\mathcal{I}_1| \lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + n \log m \sum_j D_j E_j + m + \left(\sum_j D_j E_j \right)^4. \quad (73')$$

This concludes the proof of Lemma 39, modulo the proofs of Lemmas 41 and 43. \square

3.4.1. Szemerédi-Trotter on a domain.

Proof of Lemma 41. Since all of the quantities we wish to consider are invariant under homeomorphism, without loss of generality we can assume U is an open subset of \mathbb{R}^2 . Now, replace each curve $\gamma \in A$ with a slightly “shrunk” curve γ' , so that $\partial(\gamma')$ does not meet any point from B nor any curve from A . If A' denotes the set of shrunk curves, then $|I(A', B)| \geq |I(A, B)| - 2|A|$. Delete from A' those curves γ' that are incident to fewer

than 2 points from B , and denote the resulting set of curves A'' . Then $|I(A'', B)| \geq |I(A, B)| - 4|A|$.

Consider the drawing H of the multi-graph whose vertices are the points of B and where two vertices are connected by an edge if the two corresponding points are joined by a curve from A'' . Then H is an admissible graph drawing. The multigraph H need not be a graph, since two vertices can be connected by several edges. However, the maximum edge multiplicity of H is bounded by some constant C_0 . Furthermore, $\mathcal{E}(H) \geq \frac{1}{2}|I(A'', B)|$, so by Theorem 15,

$$|I(A, B)| \lesssim |B|^{2/3} \mathcal{C}(U)^{1/3} + |B| + 4|A|. \quad \square$$

The proof of Lemma 41' is very similar to Pach and Sharir's proof in [21] of the Szemerédi-Trotter theorem for curves with k degrees of freedom, but there are a few key places where the proof differs (and the final conclusion of Lemma 41' is slightly weaker). For completeness, we have included a sketch of Lemma 41' in Appendix B.

Remark 46. An earlier result similar to Lemma 41' was proved by Pach and Sharir in [22]. There, the authors required that the curves in A be algebraic curves defined by k real parameters, i.e. there is an assignment from these k parameters to the curves such that any choice of parameters defines at most a constant number of curves, and the dependence of the curves on the parameters is algebraic of low degree. While the curves we are dealing with are algebraic and are defined by k parameters, the dependence of the curves on the parameters is of high degree, so the result from [22] is not sufficient for our purposes.

3.4.2. $\mathbb{R}\text{-Alg} \longrightarrow C^0$.

Proof of Lemma 43. We shall begin by constructing the “error set” \mathcal{I}' :

$$\mathcal{I}' = \{(p, S) \in \mathcal{I} : p \in Y_{\text{sing}}, p \text{ lies on a 1-dimensional component of } S \cap Y_{\text{sing}}\}. \quad (74)$$

First, note that Y_{sing} is an algebraic curve, and

$$\deg(Y_{\text{sing}}) \leq \left(\sum_j D_j E_j \right)^2.$$

Now, if $p \in (Y_{\text{sing}})_{\text{smooth}}$ and there exist two distinct surfaces $S, S' \in \mathcal{S}$ such that both (p, S) and (p, S') are in \mathcal{I}' , then $S \cap S'$ must contain a 1-dimensional component of Y_{sing} , so in particular $S \cap S'$ is not a discrete set, which contradicts the fact that \mathcal{S} is a good collection of surfaces. Thus for each $p \in (Y_{\text{sing}})_{\text{smooth}}$, there exists at most one $S \in \mathcal{S}$ for which $(p, S) \in \mathcal{I}'$. Thus it suffices to control

$$\sum_{p \in (Y_{\text{sing}})_{\text{sing}}} \# \text{ of curves of } Y_{\text{sing}} \text{ passing through } p. \quad (75)$$

However, (75) is bounded by the degree of $(Y_{\text{sing}})_{\text{sing}}$, which is

$$O\left(\left(\sum_j D_j E_j\right)^4\right).$$

This establishes (64).

Now for each $S \in \mathcal{S}$, we shall partition the curves of $S \cap Y$ that do not lie in Y_{sing} into classes $\Gamma_{S,j}$. Write $S \cap Y$ as a union of irreducible real curves and isolated points. To each real curve γ , either $\gamma \subset Y_{\text{sing}}$, or there is a unique index j for which every 1-dimensional connected component of γ lies in Y_j (note that γ may have 0-dimensional components, which need not lie in Y_j , but by our assumptions on \mathcal{I} no incidences can occur on these components). If $\gamma \not\subset Y_{\text{sing}}$, place γ in $\Gamma_{S,j}$ for this choice of j . Similarly, partition \mathcal{P} into collections $\{\mathcal{P}_j\}$, where $p \in \mathcal{P}_j$ if j is the minimal index for which $p \in Y_j$. Then if $p \in \mathcal{P}$, $S \in \mathcal{S}$, $(p, S) \in \mathcal{I} \setminus \mathcal{I}'$, then there is a unique index j such that $p \in \mathcal{P}_j$ and there exists a curve $\gamma \in \Gamma_{S,j}$ with $p \in \gamma$. The only way this can fail to be the case is if $p \in \mathcal{P}_j$ and every irreducible curve $\gamma \subset S \cap Y$ containing p lies in $\Gamma_{S,j'}$ for some index $j' \neq j$. But this would imply that p is an isolated point of $S \cap Y_j$, and by assumption \mathcal{I} does not contain any pairs (S, p) for which this can occur.

Thus

$$\mathcal{I} \setminus \mathcal{I}' \subset \bigcup_j \mathcal{I} \cap \{(p, S) : p \in \gamma \text{ for some } \gamma \in \Gamma_{S,j}\}. \quad (76)$$

We wish to consider each variety Y_j and the associated points and curves on that variety separately. However, before we do this we need to ensure that the same intersection $S \cap S'$ does not get counted as two distinct “crossings” on two different varieties (say Y_j and $Y_{j'}$). To prevent this from happening, we will keep track of the points where such an accident might occur. To this end, for each $S \in \mathcal{S}$ and each index j let

$$\begin{aligned} \Xi_{S,j} = \bigcup_{\gamma \in \Gamma_{S,j}} \{x \in \gamma : \text{there exists an index } j' \neq j \text{ such that} \\ x \text{ is a discrete point of } \gamma \cap Y_{j'}\}. \end{aligned} \quad (77)$$

Definition 47. Let $\gamma \in \Gamma_{S,j}$, and let $x \in \mathbb{R}^4$. Then we define $\text{mult}(\gamma, x)$ as follows. Let γ' be a generic projection of γ onto \mathbb{R}^2 , let x' be the image of x under the same projection, and let g be the squarefree polynomial with $\mathbf{Z}(g) = \gamma'$. Then $\text{mult}(\gamma, x)$ is the order of vanishing of g at x' . In particular, if $x \notin \gamma$ then $\text{mult}(\gamma, x) = 0$.

By the same arguments used above to obtain (68),

$$\sum_{x \in \Xi_{S,j}} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x) \leq E_j \sum_{\gamma \in \Gamma_{S,j}} \deg \gamma. \quad (78)$$

On the other hand, we have

$$\sum_j \sum_{S, S'} \sum_{\substack{\gamma \in \Gamma_{S,j} \\ \gamma' \in \Gamma_{S',j}}} |(\gamma \setminus \Xi_{S,j}) \cap (\gamma' \setminus \Xi_{S',j})| \leq \sum_{S, S'} |S \cap S'|. \quad (79)$$

Indeed, the only way (79) could fail is if the following occurs: there exist surfaces S and S' , a point $x \in S \cap S'$, indices j and \tilde{j} , and curves $\gamma \in \Gamma_{S,j}$, $\tilde{\gamma} \in \Gamma_{S,\tilde{j}}$, $\gamma' \in \Gamma_{S',j}$, and $\tilde{\gamma}' \in \Gamma_{S',\tilde{j}}$ such that

$$x \in \gamma \cap \gamma', \quad x \in \tilde{\gamma} \cap \tilde{\gamma}', \quad (80)$$

and

$$x \notin \Xi_{S,j} \cup \Xi_{S,\tilde{j}} \cup \Xi_{S',j} \cup \Xi_{S',\tilde{j}}. \quad (81)$$

But if (80) holds then x is an isolated point of $\gamma \cap Y_{\tilde{j}}$, so $x \in \Xi_{S,j}$ and thus (81) fails. This establishes (79).

Thus, we can consider each variety Y_j and its associated point and curve sets $\mathcal{P}_j, \{\Gamma_{S,j}\}_{S \in \mathcal{S}}$ individually. We are reduced to proving the following lemma.

Lemma 48. *Suppose we are given the following data:*

- A 2-dimensional real variety of the form $Y_j = Z_j \cap W_j$ with $Z_j = \mathbf{Z}(P_j)$, $W_j = \mathbf{Z}(Q_j)$, $\deg(P_j) = D_j$, $\deg(Q_j) = E_j$.
- A collection of points \mathcal{P} , a collection of good surfaces \mathcal{S} , and for each $S \in \mathcal{S}$, a collection $\Gamma_{S,j}$ of irreducible curves in $S \cap Y_j$ such that for each $\gamma \in \Gamma_{S,j}$, $\gamma \not\subset (Y_j)_{\text{sing}}$.
- A collection of k -admissible incidences $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$, such that for each $(p, S) \in \mathcal{I}$, there exists $\gamma \in \Gamma_{S,j}$ with $p \in \gamma$.
- For each $S \in \mathcal{S}$, a finite collection of “bad” points Ξ_S .

Then there exists an incidence model $\mathcal{M}_j = \{(U_i, A_i, B_i, \iota_i)\}_{i=1}^M$ satisfying:

- (i) For each i , A_i is a collection of simple open curves, B_i is a collection of points, U_i is homeomorphic to a open subset of \mathbb{R}^2 , and $\iota_i: U_i \hookrightarrow \mathbb{R}^4$ is an embedding.

(ii)

$$\begin{aligned} & \sum_i \sum_{\alpha, \alpha' \in A_i} |\alpha \cap \alpha'| \\ & \leq \sum_{S, S' \in \mathcal{S}} \left| \left(\bigcup_{\gamma \in \Gamma_{S,j}} \gamma \setminus \Xi_S \right) \cap \left(\bigcup_{\gamma' \in \Gamma_{S',j}} \gamma' \setminus \Xi_{S'} \right) \right|, \end{aligned} \quad (82)$$

i.e. the number of crossings between pairs of curves in the incidence model is controlled by the number of crossings of the curves from $\Gamma_{S,j}$ and $\Gamma_{S',j}$ that do not occur on the “bad” sets Ξ_S or $\Xi_{S'}$, as S and S' range over all pairs of surfaces.

- (iii) If $p \in \mathcal{P}$, $S \in \mathcal{S}$, and $\gamma \in \Gamma_{S,j}$, then there exists some index i , some $\tilde{p} \in B_i$, and some $\alpha \in A_i$ such that $\iota_i(\tilde{p}) = p$, $\alpha = \iota_i^{-1}(S)$, and $\tilde{p} \in \alpha$, i.e. the incidence model counts curve-point incidences.

(iv) For each i , the curves in A_i have k degrees of freedom (in the sense of Item (iii) from Lemma 43).

$$(v) \quad \sum_i |A_i| \lesssim (D_j^2 + D_j E_j) n + \sum_{S \in \mathcal{S}} \sum_{x \in X_{i_S}} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x), \quad (83)$$

where $\text{mult}(\gamma, x)$ is as defined in Definition 47.

Once we have Lemma 48, we can prove Lemma 43 as follows. For each index j , apply Lemma 48 to each collection $(Y_j, \mathcal{S}, \{\Gamma_{S,j}\}_{S \in \mathcal{S}}, \mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}), \{\Xi_{S,j}\})$, and denote the resulting incidence model $\mathcal{M}_j = \{(U_i^{(j)}, A_i^{(j)}, B_i^{(j)}, \iota_i^{(j)})\}$. From (79) and (82), we have

$$\sum_j \sum_i \sum_{\alpha, \alpha' \in A_i^{(j)}} |\alpha \cap \alpha'| \lesssim |\mathcal{S}|^2. \quad (84)$$

If $(p, S) \in \mathcal{I}$, then as noted above there exists a unique index j and a curve $\gamma \in \Gamma_{S,j}$ with $p \in \mathcal{P}_j$ and $p \in \gamma$. But then by Property (iii), there exists some index i , some $\tilde{p} \in B_i^{(j)}$, some $\alpha \in A_i^{(j)}$ such that $p \in \iota_i^{(j)}(\tilde{\alpha})$.

Finally,

$$\begin{aligned} \sum_j \sum_i |A_i^{(j)}| &\lesssim n \sum_j (D_j^2 + D_j E_j) + \sum_{S \in \mathcal{S}} |\Xi_S^{(j)}| \\ &\lesssim n \sum_j D_j E_j, \end{aligned} \quad (85)$$

where on the final line we used the fact that $D_j \lesssim E_j$.

Thus the incidence model $\mathcal{M} = \bigcup_j \mathcal{M}_j$ verifies the requirements of Lemma 43. This concludes the proof of Lemma 43, modulo the proof of Lemma 48. \square

Proof of Lemma 48. We shall select a very large ball B_0 containing all of the points from \mathcal{P} , and we shall decompose $B_0 \cap (Y_j)_{\text{smooth}}$ into a union of 2-dimensional C^∞ manifolds. The manifolds will have the property that for a suitably chosen 2-plane

$$\Pi_0 = \langle e_1, e_2 \rangle, \quad (86)$$

any affine translate of Π_0 will intersect a given manifold at most once (such manifolds are called “monotone” in the computational geometry literature).

Indeed, let

$$X_j = \{z \in (Y_j)_{\text{smooth}} : \dim(\Pi_0 \cap T_z(Y_j)) \geq 1\}. \quad (87)$$

If our choices for e_1 and e_2 in (86) are generic, then X_j may be empty or it may be a union of isolated points and 1-dimensional curves. We can see that each (necessarily smooth) connected component of $(Y_j)_{\text{smooth}} \setminus X_j$ is monotone: First, let $\pi: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a generic projection in the direction of some vector $v \in \Pi_0$, so $\pi(\Pi_0) \subset \mathbb{R}^3$ is a line passing through the origin (see Figure 1). Let e be a vector so that $\langle e \rangle = \pi(\Pi)$. Then if U is a (necessarily

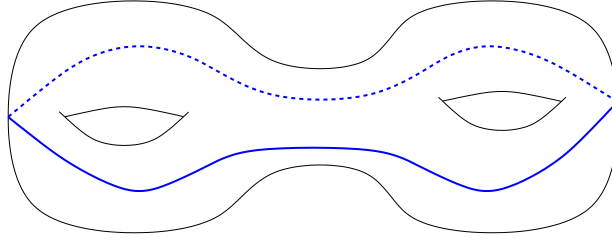


FIGURE 1. Here, Y_j is a 2-torus in \mathbb{R}^4 . In the figure, we have projected Y into \mathbb{R}^3 with a projection π chosen so that $\pi(\Pi_0)$ is a vertical line passing through the origin. The set X_j is denoted by the blue line.

bounded) connected component of $B_0 \cap (Y_j)_{\text{smooth}} \setminus X_j$ then $\pi(U)$ is also bounded and connected. It suffices to show that for any $z \in \pi(U)$, z is a smooth point of $\pi(U)$, and the line $z + \langle e \rangle$ meets $\pi(U)$ solely at the point z . First, suppose $\pi(U)$ is not smooth, and let z be a singular point. Then since U is smooth and π is a local diffeomorphism in a neighborhood of each pre-image point of $\pi^{-1}(z)$, in a small neighborhood of z , $\pi(U)$ is a union of distinct, smooth 2-manifolds, each of which contains z . Thus, we can find a nearby point z' for which $z' + \langle e \rangle$ meets $\pi(U)$ in at least two distinct points, call them z'_1 and z'_2 . Now, let β be a smooth path in U connecting a pre-image of z'_1 to a pre-image of z'_2 . Then $\pi(\beta)$ is a smooth curve in $\pi(U)$, and $d(\pi(\beta))$ always lies in $TC_g(U)$, the geometric tangent cone of U . But $TC_g(U)$ never contains $\langle e \rangle$, which is a contradiction. Thus each compact connected component of $(Y_j)_{\text{smooth}} \setminus X_j$ is monotone.

We must now count how frequently a curve $\gamma \in \Gamma_{S,j}$ intersects $(Y_j)_{\text{sing}} \cup X_j$. This will be done in the next two lemmas.

Lemma 49. *Let $\mathcal{P}, \mathcal{S}, Y_j, Z_j, W_j, P_j, Q_j$ and $\{\Gamma_{S,j}\}_{S \in \mathcal{S}}$ be as in the statement of Lemma 48. Select $\gamma \in \Gamma_{S,j}$ and let X_j be as in (87). Then*

$$|\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| \lesssim \deg \gamma \cdot (D_j + E_j). \quad (88)$$

Proof. First, note that if our choices of e_1 and e_2 in (86) are generic, then for each curve $\gamma \in \bigcup_S \Gamma_{S,j}$, we have that $X_j \cap \gamma$ will be a discrete set of points. Thus, we can guarantee the following things:

- (1) Every point of $X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$ is a smooth point of γ and a smooth point of X_j . In particular, no point of $X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$ is an isolated point of X_j .
- (2) At every point $z \in X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$, we have the following property: let $B \subset \mathbb{C}^4$ be a sufficiently small ball centered at z^* , so that if $Gr(4, 2; \mathbb{C})$ is the Grassmanian of (complex) 2-planes in \mathbb{C}^4 , then the tangent bundle

$$T(B^* \cap Z_j^* \cap W_j^*) = \{(z, \Pi) \in \mathbb{C}^4 \times Gr(4, 2; \mathbb{C}) : \\ z \in B \cap (Y_j)_{\text{smooth}}, \Pi = T_z(Y_j)\}$$

is a smooth 2-(complex)-dimensional submanifold of $\mathbb{C}^4 \times Gr(4, 2; \mathbb{C})$.
Let

$$J = \{(\Pi, z) \in \mathbb{C}^4 \times Gr(4, 2; \mathbb{C}) : \dim(\Pi \cap \Pi_0) \geq 1\}. \quad (89)$$

Then if \tilde{z} is the image of z in $\mathbb{C}^4 \times Gr(4, 2; \mathbb{C})$, we have that $T(B \cap Z_j^* \cap W_j^*)$ and J intersect transversely at \tilde{z} , so the intersection is a smooth curve in $\mathbb{C}^4 \times Gr(4, 2; \mathbb{C})$. Thus in particular, if we apply a small C^∞ perturbation to the surface $B \cap Z_j^* \cap W_j^*$, then the image of the perturbed surface in $\mathbb{C}^4 \times Gr(4, 2; \mathbb{C})$ will still intersect J transversely.

First, observe: If (P_j, Q_j) is a reduced ideal, i.e. if $\dim(\frac{\nabla P_j}{\nabla Q_j}) = 2$ on $(Y_j)_{\text{smooth}}$, then

$$(Y_j)_{\text{sing}} \cup X_j = Y_j \cap \mathbf{Z}(\Psi(P_j, Q_j; \cdot)), \quad (90)$$

where

$$\Psi(P_j, Q_j; z) = \det \begin{bmatrix} e_1 \\ e_2 \\ \nabla P_j \\ \nabla Q_j \end{bmatrix} (z). \quad (91)$$

Then, to compute $|\gamma \cap (X_j \cup (Y_j)_{\text{sing}})|$, it suffices to count the number of intersection points in $\gamma \cap \mathbf{Z}(\Psi(P_j, Q_j; \cdot))$, and this is a (set-theoretic) complete intersection. We are working over \mathbb{R} , so we cannot appeal directly to Bézout's theorem, but we can use arguments similar to those used to obtain (68).

In order to make this argument work, we will need to perturb P_j and Q_j to make (P_j, Q_j) a reduced ideal. Doing so will cause Y_j to “split” (in a small neighborhood of a smooth point) into several sheets, and (locally) there will be one or more copy or copies of γ on each sheet. Through careful counting, we can recover the above result.

We will need several elementary results from intersection theory. Further details can be found in standard textbooks such as [10, 12]. Throughout this section, we will choose an embedding of \mathbb{C}^4 into \mathbb{CP}^4 . This will allow us to identify points in \mathbb{C}^4 with those in \mathbb{CP}^4 , and to identify (complex, affine) polynomials with homogeneous polynomials.

Definition 50. If $f \in \mathbb{C}[x_1, \dots, x_4]$, we will let I_f denote the projective ideal generated by f (here as elsewhere, f is identified with its image under the above embedding from \mathbb{C}^4 into \mathbb{CP}^4). If $f \in \mathbb{C}[x_1, \dots, x_4]$, let $\mathbf{Z}^*(f)$ be the zero set of f (either in \mathbb{C}^4 or in \mathbb{CP}^4 , depending on context). If $x \in \mathbb{C}^4$, then $\mathcal{O}_{\mathbb{CP}^4, x}$ is the local ring obtained by localizing \mathbb{CP}^4 at the point x (again, we have identified x with its image in \mathbb{CP}^4). If f_1, f_2, \dots, f_k are polynomials, we say that f_1, \dots, f_k intersect properly if $\text{codim}(\mathbf{Z}^*(f_1) \cap \dots \cap \mathbf{Z}^*(f_k)) = k$.

If f_1, \dots, f_k intersect properly, and V is an irreducible component of $\mathbf{Z}^*(f_1) \cap \dots \cap \mathbf{Z}^*(f_k)$, then we define

$$\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V) = \dim \mathcal{O}_{\mathbb{CP}^4, x} / (I_{f_1} + \dots + I_{f_k}),$$

where x is a generic point of V .

Remark 51. This definition of multiplicity has the following useful geometric property. Let f_1, \dots, f_k intersect properly and let V be an irreducible component of $\mathbf{Z}^*(f_1) \cap \dots \cap \mathbf{Z}^*(f_k)$. Select a generic point $x \in V$ and a generic hyperplane H of dimension $\text{codim}(V)$, such that $x \in V \cap H$. Then we can find a small ball $B \subset \mathbb{C}^4$ containing x such that x is the only point in $B \cap H \cap V$. Now, let $f'_1 = f_1 + \epsilon_1, \dots, f'_k = f_k + \epsilon_k$, where $\epsilon_1, \dots, \epsilon_k$ are generic real numbers with sufficiently small magnitude. Then $B \cap V \cap \mathbf{Z}^*(f'_1) \cap \dots \cap \mathbf{Z}^*(f'_k)$ is a union of $\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V)$ points: if B is a small ball centered around a generic point of V , then if we perturb f_1, \dots, f_k , the intersection “splits” into $\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V)$ sheets.

Let S be the surface and let $\gamma \in \Gamma_{S,j}$ be the curve in the statement of Lemma 49. Let

$$S^\dagger = \bigcup_{x \in S} (x + \langle e \rangle), \quad (92)$$

where e is a generic vector in \mathbb{R}^4 . We can verify that S^\dagger is an irreducible 3-dimensional variety of bounded degree: in short, select a rotation of \mathbb{R}^4 so that e is the x_1 -direction. By the Tarski-Seidenberg theorem (see e.g. [5]), the projection $\pi_{x_1}(S)$ is a bounded degree algebraic variety, with ideal $I \subset \mathbb{R}[x_2, x_3, x_4]$. Let $S^\dagger = \mathbf{Z}(I^\dagger)$, where $I^\dagger \in \mathbb{R}[x_1, \dots, x_4]$ is the canonical embedding of I into $\mathbb{R}[x_1, \dots, x_4]$. S^\dagger has codimension 1, so we can write $S^\dagger = \mathbf{Z}(f_S)$ for some polynomial f_S that generates a real ideal.

Now, $S^\dagger \cap Y_j$ is a 1-dimensional curve, and thus $S^\dagger \cap Y_j$ is a proper intersection. We also have $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ is a 1-dimensional (complex) curve and so $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ is a proper intersection. Since $\gamma \subset S \cap Y_j$ we also have $\gamma \subset S^\dagger \cap Y_j$, so there exists an irreducible component $\gamma^\dagger \subset S^\dagger \cap Y_j$ such that (as sets) $\gamma^\dagger = \gamma$. We define $(\gamma^\dagger)^* \subset (S^\dagger)^* \cap Z_j^* \cap W_j^*$ similarly. By Proposition 35, $(\gamma^\dagger)^*$ is irreducible. Thus $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ is well-defined. We define

$$\text{mult}(Z_j^*, W_j^*; (\gamma^\dagger)^*) = \dim \mathcal{O}_{\mathbb{P}^4, x} / (I_{P_j} + I_{Q_j}), \quad (93)$$

where x is a generic point of $(\gamma^\dagger)^*$. Since $(\gamma^\dagger)^*$ lies generically in $(Z_j^* \cap W_j^*)_{\text{smooth}}$, there exists a unique irreducible component V of $(Z_j^* \cap W_j^*)_{\text{smooth}}$ that contains $(\gamma^\dagger)^*$, and $\text{mult}(Z_j^*, W_j^*; \gamma) = \text{mult}(Z_j^*, W_j^*; V)$, where the latter multiplicity is given by Definition 50.

We will need a rigorous way to show that if two manifolds intersect transversely, then slight perturbations of the manifolds still intersect transversely. The following definition will aid in this process.

Definition 52. Let U and U' be smooth manifolds in some ambient space X (we will have either $X = \mathbb{R}^4$, $X = \mathbb{C}^4$, or $X = \mathbb{C}^4 \times Gr(4, 2; \mathbb{C})$). We say U and U' are ϵ -close if there exists a diffeomorphism $\psi: X \rightarrow X$ sending U to U' , and $\|\psi - \text{Id}\|_{C^0} < \epsilon$. If U' is ϵ -close to U , we say U' is an ϵ -perturbation of U .

Let $P'_j = P_j - \epsilon_1$, $Q'_j = Q_j - \epsilon_2$, where ϵ_1, ϵ_2 are chosen generically from the interval $(0, \epsilon)$; ϵ will be chosen later. Then (P'_j, Q'_j) is a reduced ideal. We claim: If z is a smooth point of γ (and thus z^* is a smooth point of $(\gamma^\dagger)^*$), a smooth point of Y_j , and a smooth point of $S^\dagger \cap Y_j$, then if $B \subset \mathbb{C}^4$ is a sufficiently small ball centered at x^* , then:

- $B \cap (Z'_j)^* \cap (W'_j)^*$ is a union of $\text{mult}(Z_j^*, W_j^*; (\gamma^\dagger)^*)$ smooth disjoint 2-manifolds, and each of these 2-manifolds is a $o(1; \epsilon)$ -perturbation of $B \cap Z_j^* \cap W_j^*$.
- $B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ is a union of $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ smooth curves, and each of these curves is a $o(1; \epsilon)$ -perturbation of $B \cap (\gamma^\dagger)^*$. These curves lie on the various connected components of $B \cap (Z'_j)^* \cap (W'_j)^*$.

The key observation is the following.

Lemma 53. *Select $z \in \gamma \cap X_j \cap (Y_j)_{\text{smooth}}$ and let $\rho > 0$. Let B be the ball centered at z of radius ρ . Then provided ϵ is sufficiently small (depending on ρ), we have*

$$\begin{aligned} & |B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^* \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))| \\ & \geq \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*). \end{aligned} \quad (94)$$

Proof. The idea is to show that each of the $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ 1-dimensional curves in $B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ intersects some 1-dimensional curve from $B \cap (Z'_j)^* \cap (W'_j)^* \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$. We expect this to happen because the (unperturbed) curves intersect transversely, and the perturbation is very small.

Indeed, let $\zeta \subset B$ be a simple curve from $B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$. Then since $\gamma \cap B$ is smooth, ζ is a $o(1; \epsilon)$ -perturbation of $\gamma \cap B$, in the sense of Definition 52. Let $U \subset B \cap (Z'_j)^* \cap (W'_j)^*$ be the smooth 2-manifold containing ζ . Then since $B \cap X_j$ is smooth, $U \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$ is a $o(1; \epsilon)$ perturbation of $B \cap X_j^*$. Now, z is one of finitely many intersection points of γ and X_j , and each of these intersections are transverse. Thus if we select ϵ sufficiently small (depending on both the transversality of the intersection and ρ), then ζ and $U \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$ must intersect. Thus in particular, ζ intersects $B \cap (Z'_j)^* \cap (W'_j)^* \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$. But there are $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ such curves ζ . This establishes the lemma. \square

We claim: if ϵ is sufficiently small, then there exists a curve $\zeta_\gamma \subset (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ which is the “image” of $(\gamma^\dagger)^*$ under the above perturbation. Indeed, we can select a small constant c so that if $G \subset (\gamma^\dagger)^*$ is the set of points that are distance at least $2c$ from any point of $Z_j^* \cap W_j^* \cap (S^\dagger)^*$ that does not lie in $(\gamma^\dagger)^*$, then G contains an open interval. Let $z \in G \cap (\gamma^\dagger)^*_{\text{smooth}} \cap (Z_j^* \cap W_j^*)_{\text{smooth}}$ be a point contained in the relative interior of this open interval, and let B_1 be a small ball centered at z . Then if we select

ϵ sufficiently small, then $B_1 \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ is a union of curves, each of which is a c -perturbation of $B_1 \cap (\gamma^\dagger)^*$. Let ζ_γ be the smallest algebraic set containing $B_1 \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$. We have $\zeta_\gamma \subset (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$. ζ_γ corresponds to the intuitive notation of the “image” of $(\gamma^\dagger)^*$ under the perturbation.

We can now bound the degree of ζ_γ . Let $H \subset \mathbb{C}^4$ be a generic 3-plane. Then $|H \cap \gamma| = \deg \gamma$. But by the definition of multiplicity above, to each point $x \in H \cap \gamma$, we can find a small ball B centered at x such that

$$B \cap \zeta_\gamma \cap H = \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*).$$

Thus

$$\deg \zeta_\gamma = \deg \gamma \cdot \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*). \quad (95)$$

Lemma 53 can be rephrased as the statement

$$\begin{aligned} & |\zeta_\gamma \cap \mathbf{Z}^*(\Phi(P'_j, Q'_j; \cdot))| \\ & \geq |\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| \cdot \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*), \end{aligned} \quad (96)$$

i.e.

$$|\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| \leq \frac{|\zeta_\gamma \cap \mathbf{Z}^*(\Phi(P'_j, Q'_j; \cdot))|}{\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)}. \quad (97)$$

But this and (95) imply that

$$\begin{aligned} |\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| & \leq \deg \gamma \cdot \deg \Phi(P'_j, Q'_j; \cdot) \\ & \lesssim \deg \gamma \cdot (D_j + E_j). \end{aligned} \quad (98)$$

This concludes the proof of Lemma 49. \square

Lemma 54. *Let \mathcal{S} , Y_j , Z_j , W_j , P_j , Q_j and $\{\Gamma_{S,j}\}_{S \in \mathcal{S}}$ be as in the statement of Lemma 48, and let $S \in \mathcal{S}$. Then*

$$\sum_{\gamma \in \Gamma_{S,j}} \sum_{x \in \gamma \cap (Y_j)_{\text{sing}}} \text{mult}(\gamma, x) \lesssim D_j E_j + D_j^2. \quad (99)$$

Proof. Let S^\dagger be as defined in (92). For each $\gamma \in \Gamma_{S,j}$, let γ^\dagger and $(\gamma^\dagger)^*$ be defined as above. Then if $x \in \gamma \cap (Y_j)_{\text{sing}}$, there must be smooth points of Y_j in every (Euclidean) neighborhood of x ($\gamma \in \Gamma_{S,j}$ implies γ lies generically on $(Y_j)_{\text{smooth}}$). Thus x^* is a singular point of $Z_j^* \cap W_j^*$, so

$$\dim \mathcal{O}_{\mathbb{CP}^4, x} / (I_{f_S} + I_{P_j} + I_{Q_j}) > \text{mult}((S^\dagger)^*, Z_j^*, W_j^*; (\gamma^\dagger)^*). \quad (100)$$

From this we can conclude that x^* is a singular point of the (not necessarily irreducible) curve $(S^\dagger)^* \cap Z_j^* \cap W_j^*$. Furthermore, the number of intersection points of $(\gamma^\dagger)^*$ with $(Z_j^* \cap W_j^*)_{\text{sing}}$ (counting multiplicity) as γ ranges over all curves in $\Gamma_{S,j}$ is controlled by the number of singular points of $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ that occur on S^* (again, counting multiplicity). Let $\zeta \subset (S^\dagger)^* \cap Z_j^* \cap W_j^*$ be the union of all irreducible curves in $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ that are not contained in S^* . Then the number of singular points of $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ that occur on S^* is at most the number of singular points of $S^* \cap Z_j^* \cap W_j^*$ plus the

number of intersection points of ζ with S^* (again, counting multiplicity). The former quantity is at most D_j^2 , and the latter is at most $D_j E_j$. \square

We can now complete the proof of Lemma 48. Fix a choice of $S \in \mathcal{S}$. For each $\gamma \in \Gamma_{S,j}$, Delete the points that lie in Ξ_S , the points of $\gamma \cap (Y_j)_{\text{sing}}$, the points of $\gamma \cap X_j$, and the points that lie on some $\gamma' \neq \gamma$. We can verify that after these points have been removed, the remaining set is a disjoint union of connected 1-dimensional manifolds. Furthermore, the number of 1-manifolds is

$$\begin{aligned} & O\left((D_j + E_j) \sum_{\gamma \in \Gamma_{S,j}} \deg \gamma + D_j^2 + D_j E_j + \left(\sum_{\gamma \in \Gamma_{S,j}} \deg \gamma\right)^2\right. \\ & \quad \left.+ \sum_{x \in \Xi_S} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x)\right) \\ & = O\left(D_j^2 + D_j E_j + \sum_{x \in \Xi_S} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x)\right). \end{aligned} \quad (101)$$

Indeed, every time a point x is removed, the number of manifolds can increase by at most $\sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x)$. If $x \in \gamma \cap (Y_j)_{\text{smooth}} \cap X_j$, then $\text{mult}(\gamma, x) = 1$, so the number of curves added by removing points of this type is at most $O\left((D_j + E_j) \sum_{\gamma \in \Gamma_{S,j}} \deg \gamma\right) = O(D_j^2 + D_j E_j)$. We can apply similar bounds to control the number of manifolds that are added when the other types of points are excised.

Now, each of these 1-manifolds is homeomorphic to either the interval $(0, 1)$ or the circle S^1 . For those manifolds that are homeomorphic to circles, remove two points at random to obtain two curves homeomorphic to $(0, 1)$. Our new collection of simple curves has the same cardinality (up to a factor of two), and each curve lies entirely within a single component of $(Y_j)_{\text{smooth}} \setminus X_j$. Let $\{U_i\}$ be the connected components of $(Y_j)_{\text{smooth}} \setminus X_j$, and let A_i consist of those simple curves lying in U_i , as γ ranges over $\bigcup_{S \in \mathcal{S}} \Gamma_{S,j}$. Let $B_i = U_i \cap \mathcal{P}_j$ and let $\iota_i: U_i \hookrightarrow \mathbb{R}^4$ be the canonical embedding of U_i into \mathbb{R}^4 .

From (101) we have

$$\sum_i |A_i| \lesssim D_j^2 + D_j E_j + \sum_{S \in \mathcal{S}} \sum_{x \in \Xi_S} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x), \quad (102)$$

so Requirement (v) is satisfied. In order to verify the remaining requirements we shall introduce some notation

Definition 55. Let $\alpha \in A_i$ for some index i . Then there exists a unique surface $S_0 \in \mathcal{S}$ and a unique curve $\gamma_0 \in \Gamma_{S_0,j}$ such that $\alpha \subset \iota_i^{-1}(\gamma)$. We will define $S(\alpha)$ to be S_0 and we will define $\gamma(\alpha)$ to be γ_0 .

We can verify that $\mathcal{P}_j = \bigcup_i B_i$, i.e. that $X_j \cap \mathcal{P}_i = \emptyset$, since the vectors e_1 and e_2 from (86) were chosen generically. Recall as well that by assumption, $\mathcal{P}_j \cap (Y_j)_{\text{sing}} = \emptyset$. If $\gamma \in \Gamma_{S,j}$, $p \in \mathcal{P}$, $p \in \gamma$, then we can immediately verify

that either there exists an index i and a curve $\alpha \in A_i$ with $\gamma(\alpha) = \gamma$ and $p \in \partial(\iota_i(\alpha))$, or there exists an index i , a curve $\alpha \in A_i$, and a point \tilde{p} in B_i with $\iota_i(\tilde{p}) = p$, $\gamma(\alpha) = \gamma$, and $\tilde{p} \in \alpha$. Thus Requirement (iii) is satisfied.

To verify Requirement (ii), note that if $\alpha, \alpha' \in A_i$ and $x \in \alpha \cap \alpha'$, Then $\iota_i(x) \in \iota_i(\alpha) \cap \iota_i(\alpha)'$, and $x \notin \Xi_{S(\alpha)} \cup \Xi_{S(\alpha')}$. Furthermore, (α, α') are the unique pair of curves in $\{A_i\}$ corresponding to the triple $(\iota_i(x), \gamma(\alpha), \gamma(\alpha'))$.

To obtain Requirement (iv), we shall apply a slight perturbation to the curves of $\{A_i\}$ as follows: for each curve $\alpha \in A_i$ and point $p \in B_i$, if $p \in \alpha$ but $(S(\alpha), \iota(p)) \notin \mathcal{I}$, modify α in a small neighborhood of p so that $p \notin \alpha$. We can always do this in such a way that the number of crossings between α and the other curves is not affected, and α remains unchanged in a small neighborhood of every point $p' \in B_i$ distinct from p . After this perturbation has been performed for every curve, then the collection $\{(A_i, B_i)\}$ satisfies Requirement (iv). Indeed, since \mathcal{I} is k -admissible, if there exists an index i , a collection of k points $p_1, \dots, p_k \in B_i$, and a collection of $C_0 + 1$ curves in A_i such that each curve is incident to each of p_1, \dots, p_k , then there must exist two curves α, α' from this collection of curves such that $S(\alpha) = S(\alpha')$. However, this implies that $\iota_i(\alpha)$ either contains a singular point of $\gamma(\alpha)$, or it contains a point of $\gamma(\alpha')$ (if $\gamma(\alpha') \neq \gamma(\alpha)$). But by our construction of A_i , no such points may lie on any curve in A_i . Thus Requirement (iv) is satisfied.

Finally, recall that we already established Requirement (i) above (in the discussion preceding Lemma 49). \square

4. OPEN PROBLEMS AND FURTHER WORK

Theorems 9 and 9' have several obvious shortcomings, namely the fact that we do not have a uniform bound on the cardinality of a collection of 2-admissible incidences $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ which does not depend on the ratio $\log |\mathcal{P}| / \log |\mathcal{S}|$, and in addition we loose a $\log |\mathcal{P}|$ factor when bounding the cardinality of a collection of k -admissible incidences. This gives rise to our first conjecture:

Conjecture 56. Let $\mathcal{P} \subset \mathbb{R}^4$ be a collection of points, let \mathcal{S} be a good collection of surfaces, and let \mathcal{I} be a collection of k -admissible incidences. Then

$$|\mathcal{I}| \lesssim |\mathcal{P}|^{\frac{k}{2k-1}} |\mathcal{S}|^{\frac{2k-2}{2k-1}} + |\mathcal{P}| + |\mathcal{S}|. \quad (103)$$

In order to prove Conjecture 56 for the case $k = 2$, it appears that we need a way to deal with incidences between points $p \in \mathcal{P}$ and surfaces $S \in \mathcal{S}$ for which $S \cap Y_{\text{sing}}$ is a 1-dimensional curve, and p lies on $(S \cap Y_{\text{sing}})_{\text{sing}}$. Here Y is as defined in Lemma 39. For $k > 2$ we need to deal with both this problem and the logarithmic loss introduced by emulating the Pach-Sharir proof for obtaining bounds on incidences between points and curves with k degrees of freedom. It seems plausible that a cleverer modification of the Pach-Sharir techniques might remove this logarithmic loss.

We can also ask similar questions in higher dimensions, and here things appear to be more difficult. In \mathbb{R}^{2d} , we define a family of surfaces \mathcal{S} to be *good* if it satisfies the requirements from Definition 3, except in place of Requirement (i) we require that every surface in \mathcal{S} has dimension d . We define a set of incidences $\mathcal{I} \subset I(\mathcal{P}, \mathcal{S})$ to be k -admissible if it satisfies the requirements from Definition 6. We then have the following conjecture.

Conjecture 57. Let \mathcal{P} be a collection of points in \mathbb{R}^d , d even, and \mathcal{S} a collection of smooth $d/2$ -dimensional algebraic surfaces of degree $O(1)$ such that every two surfaces meet in $O(1)$ points and at most $O(1)$ surfaces pass through any collection of k points. Then

$$|\mathcal{I}| \lesssim |\mathcal{P}|^{\frac{k}{2k-1}} |\mathcal{S}|^{\frac{2k-2}{2k-1}} + |\mathcal{P}| + |\mathcal{S}|. \quad (104)$$

The bound (104) is what you would obtain if you performed a cell decomposition on the point set \mathcal{P} with a polynomial P whose degree was chosen to minimize the number of incidences inside the cells, and you ignored incidences occurring on the boundary of the cells, i.e. if you only considered Section 3.2.1 from the proof of Theorem 37. This assumption is satisfied if the points in \mathcal{P} are in general position in \mathbb{R}^d .

Conjecture 57 is an obvious generalization of a conjecture of Tóth [29, Conjecture 3], who proposed Conjecture 57 in the case where \mathcal{S} is a collection of affine d -planes. The conjecture seems plausible because a slight modification of the methods used by Solymosi and Tao in [23] yield the following slightly weaker statement:

Proposition 58. Let $\mathcal{P}, \mathcal{S}, \mathcal{I}$ be as in Conjecture 57. Then for any $\epsilon > 0$, we have

$$|\mathcal{I}| \lesssim_{\epsilon} |\mathcal{P}|^{\frac{k}{2k-1} + \epsilon} |\mathcal{S}|^{\frac{2k-2}{2k-1}} + |\mathcal{P}| + |\mathcal{S}|. \quad (105)$$

More ambitiously, we could conjecture

Conjecture 59. Let \mathcal{P} be a collection of points in \mathbb{R}^d and \mathcal{S} a collection of smooth ℓ -dimensional algebraic surfaces, $(d - \ell)|d$, of degree at most C_0 such that every $d/(d - \ell)$ surfaces meet in at most C_0 points and at most C_0 surfaces pass through any collection of k points, $k \geq d/(d - \ell)$. Then

$$\mathcal{I}(\mathcal{P}, \mathcal{S}) \lesssim |\mathcal{P}|^{\frac{k\ell}{d(k-1)+\ell}} |\mathcal{S}|^{\frac{d(k-1)}{d(k-1)+\ell}} + |\mathcal{P}| + |\mathcal{S}|. \quad (106)$$

Again, the bound in Conjecture 59 is the one we would obtain if we used the polynomial ham sandwich theorem to cut \mathbb{R}^d into cells and applied Lemma 16 to control each cell, i.e. Conjecture 59 is saying that the number of incidences between surfaces and points on the boundary of the (first level) cell decomposition is controlled by (the worst case behavior of) the number of incidences between surfaces and points in the interior of the (first level) cell decomposition.

It appears that in order to resolve these conjectures some new techniques must be developed. In particular, it appears that in order to control the number of incidences between points and ℓ -dimensional varieties, we need

to perform ℓ polynomial ham sandwich decompositions, with each successive decomposition performed on the variety defined by the previous decompositions. As ℓ increases, the number of cases to be considered increases dramatically, and certain difficulties such as the failure of the connected components of a complete intersection to themselves be complete intersection, the failure of a set-theoretic complete intersection to be a nonsingular complete intersection, etc. become increasingly problematic. A second difficulty is that the surfaces in \mathcal{S} need not form a complete intersection with the polynomial ham sandwich cuts. This was already the main difficulty for proving Theorems 9 and 9'. In our proof, however, it was only necessary to perform two ham sandwich decompositions, and we were left with a problem about points and curves on a 2-dimensional surface, so we could appeal to the crossing number inequality. In higher dimensions there is no obvious analogue of the crossing number inequality, and thus it is not obvious how to proceed.

APPENDIX A. PROOFS OF LEMMAS FROM SECTION 2

Proof of Lemma 22. Let $\mathcal{Q} = \emptyset$. Write $P = P_1, \dots, P_a$ as a product of irreducible factors. Place each irreducible factor that generates a real ideal in \mathcal{Q} . If P_j is a factor that does not generate a real ideal then consider $\nabla_v P_j$ for v a generic vector. We have $\deg \nabla_v P_j < \deg P$, and $\mathbf{Z}(P_j) \subset \mathbf{Z}(\nabla_v P_j)$. Apply the above procedure to $\nabla_v P_j$. This process will terminate after finitely many iterations. Let $\tilde{P} = \prod_{Q \in \mathcal{Q}} Q$. \square

Proof of Lemma 31. Let $p \in \mathcal{P}$, and let $H = T_p(V)$. Suppose there existed surfaces $S_1, S_2 \in \mathcal{S}$ with $(p, S), (p, S') \in \mathcal{I}$. Recall that p must be a smooth point of S and of S' . Since $S \subset V$, we have $T_p(S) \subset T_p(V) = H$. Similarly, $T_p(S') \subset H$. Since \mathcal{I} is k -good, $T_p(S) \cap T_p(S') = p$. Thus we have two affine 2-planes, $T_p(S)$ and $T_p(S')$ which meet only at the point p , but both are contained in the affine 3-plane H . This cannot occur. Thus for each point $p \in \mathcal{P}$, there can exist at most one surface $S \in \mathcal{S}$ with $(p, S) \in \mathcal{I}$. Thus

$$|\mathcal{I}| \leq |\mathcal{P}|. \quad (107)$$

\square

Proof of Lemma 32. The proof is similar to the proof of Lemma 13 in [31], so we will only give a brief sketch here. First, select a large number R so that every cell that meets S does so within the ball centered at the origin of radius R . Let $\tilde{P} = P \cdot f_B$, where $f_B(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - R^2$. Thus the number of cells of $\mathbf{Z}(P)$ that S meets is at most the number of bounded cells of $\mathbf{Z}(\tilde{P})$ that S meets. Since the property of S meeting a cell is open, we can apply a small generic translation to S and a small generic perturbation to \tilde{P} , and doing so can only increase the number of bounded cells that S meets. Now, we can find f_1, f_2 such that $S \subset \mathbf{Z}(f_1) \cap \mathbf{Z}(f_2)$, (f_1, f_2) is a reduced ideal, and f_1 and f_2 are of bounded degree. Let v be a generic vector in \mathbb{R}^4 , and let $T(x) = v \wedge \nabla f_1 \wedge \nabla f_2 \wedge \nabla \tilde{P}$. Then

$\deg T(x) \lesssim D$. Now, the number of cells of $\mathbf{Z}(\tilde{P})$ that S enters is controlled by the number of (necessarily non-singular) intersection points of S , $\mathbf{Z}(\tilde{P})$, and $\mathbf{Z}(T)$ (again, see Lemma 13 in [31] for details), and this is $O(D^2)$. \square

Proof of Lemma 33. Write $S \cap \mathbf{Z}(P)$ as a union of irreducible curves, and denote this collection of irreducible curves by Γ . By Harnack's theorem, $\bigcup_{\gamma \in \Gamma} \gamma$ can have at most $O((\deg P)^2)$ components (again, see Remark 45 for details). Now, for each irreducible curve $\gamma \in \Gamma$ and each $Q \in \mathcal{Q}$, either $\gamma^* \subset \mathbf{Z}^*(Q)$ or $|\gamma^* \cap \mathbf{Z}^*(Q)| \lesssim \deg \gamma \deg Q$, and thus $|\gamma \cap \mathbf{Z}(Q)| \lesssim \deg \gamma \deg Q$. We will call intersections of this type “important” intersections between S , Z and $\mathbf{Z}(Q)$. If $\gamma^* \subset \mathbf{Z}^*(Q)$, then since $\gamma^* \subset \mathbf{Z}^*(Q)$, $\gamma \subset \mathbf{Z}(Q)$, and thus γ does not enter *any* realizations of realizable strict sign conditions of \mathcal{Q} on Z . Now, if Ω is a realization of a realizable sign condition of \mathcal{Q} on Z , and $S \cap \Omega \neq \emptyset$, then we can associate to the pair (S, Ω) an important intersection of S , Z , and $\mathbf{Z}(Q)$ for some $Q \in \mathcal{Q}$ in such a way that every important intersection is assigned to at most 2 pairs (S, Ω) . Thus the number of realizations of realizable strict sign conditions of \mathcal{Q} on Z is at most $O(\deg P \sum_{Q \in \mathcal{Q}} \deg Q) = O(DE)$. \square

APPENDIX B. PACH-SHARIR'S BOUND FOR CURVES WITH k DEGREES OF FREEDOM

Lemma 41'. *Let U be an open subset of \mathbb{R}^2 , Let $B \subset U$ be a finite collection of points, and let A be a collection of simple open curves. Suppose there exists a constant C_0 such that the following holds:*

- *For any k points from B , there are at most C_0 curves $\alpha_1, \dots, \alpha_{C_0} \in A$ with $p_i \in \alpha_j$, $i = 1 \dots, k$ and $j = 1 \dots C_0$.*
- *Any two curves from A intersect in at most C_0 points.*
- *The boundary of each curve is disjoint from B and does not lie on any other curve.*

Let $\mathcal{C}(U)$ be the number of times two curves from B cross (see Definition 40). Then

$$\mathcal{I}(A, B) \lesssim |B|^{\frac{k}{2k-1}} \mathcal{C}(U)^{\frac{k-1}{2k-1}} + |A| \log |B| + |B|. \quad (108)$$

Proof. If we attempt to draw a graph on the region U in the same manner as in Lemma 41, we run into the difficulty that many curves can connect two vertices, and a priori we do not have a way to bound the number of such curves, so we cannot use Theorem 15 to obtain useful bounds on the number of edge crossings. To get around this problem, we will partition the points in B into roughly $\log |B|$ diadic size ranges based on the degree of the points, i.e. the number of edges meeting each point. We will use the crossing number inequality to control the number of edges in each piece of this diadic decomposition and then sum the contributions from each piece. Pach and Sharir consider the case where $\mathcal{C}(U)$ is comparable to $|A|^2$, and they are able to use some careful arguments to avoid introducing an additional $\log |B|$

factor into their bounds. Unfortunately in our case $\mathcal{C}(U)$ may be much smaller than $|A|^2$ so we cannot use their techniques to avoid the $\log|B|$ factor that appears in (108).

For $p \in B$, let $d(p)$ be the number of curves in A that contain p . Define

$$B^* = \left\{ p \in B : d(p) \leq \frac{I}{2|B|} \right\},$$

$$B_\ell = \left\{ p \in B : \frac{2^{\ell-1}I}{|B|} < d(p) \leq \frac{2^\ell I}{|B|} \right\}.$$

Let $d_\ell = 2^{\ell-1}I/|B|$ be the degree (up to a factor of 2) of the points in B_ℓ , and let $I_\ell = \mathcal{I}(B_\ell, A)$. Note that

$$|B_\ell| < 2^{1-\ell}|B|. \quad (109)$$

We have $\mathcal{I}(A, B^*) \leq \frac{1}{2}\mathcal{I}(A, B)$, and thus it suffices to control $\mathcal{I}(A, B \setminus B^*)$. Define

$$J_1 = \{\ell \geq 0 : |B_\ell|^k < \frac{|A|}{2^\ell}\},$$

$$J_2 = \{\ell \geq 0 : |B_\ell|^k \geq \frac{|A|}{2^\ell}\}.$$

We have

$$\sum_{\ell \in J_1} |B_\ell| \lesssim |A|^{1/k}.$$

Applying Lemma 16 we have

$$\mathcal{I}\left(\bigcup_{\ell \in J_1} B_\ell, A\right) \lesssim |A|. \quad (110)$$

We shall now control $\mathcal{I}(A, B_\ell)$ for $\ell \in J_2$. Let H_ℓ be the graph drawing whose vertices are the points B_ℓ , and where if two vertices both lie on a curve $\gamma \in A$ with at most $k-2$ points of B_ℓ in between them, then we connect them by the piece of γ lying between them (thus in general several edges will be drawn on top of each other, and two vertices may be connected by several different edges). Let H'_ℓ be the graph obtained from H_ℓ by erasing every edge whose multiplicity exceeds $Cd_\ell^{1-1/(k-1)}$ for some large constant C to be chosen later. We have

$$I_\ell - 4k \leq |\mathcal{E}(H_\ell)| \leq (k-1)I_\ell. \quad (111)$$

We claim that

$$I_\ell - 4k \lesssim |\mathcal{E}(H'_\ell)|. \quad (112)$$

Equation (112) is proved on pages 124–5 of [21], and the proof for us is identical.

Now, every point on a curve $\alpha \in A$ belongs to the relative interior of at most $\binom{k}{2}$ edges of H'_ℓ lying on α , so there is a subgraph $H''_\ell \subset H'_\ell$ with $|\mathcal{E}(H''_\ell)| \geq |\mathcal{E}(H'_\ell)|/\binom{k}{2}$ such that no two edges of H''_ℓ overlap, i.e. H''_ℓ is

an admissible drawing of a multigraph with maximum edge multiplicity $M_\ell = A_0 d_\ell^{1-1/(k-1)}$. Let

$$J_3 = \{\ell \in J_2 : |\mathcal{E}(H''_\ell)| \geq 5|\mathcal{V}(H''_\ell)|M_\ell\}.$$

For $\ell \in J_2 \setminus J_3$,

$$\begin{aligned} I_\ell &\lesssim \binom{k}{2} |\mathcal{E}(H''_\ell)| \\ &\lesssim |B_\ell| d_\ell^{1-1/(k-1)}, \end{aligned}$$

but since $d_\ell \leq I_\ell/|B_\ell|$, we get $I_\ell \lesssim |B_\ell|$ and thus

$$\sum_{\ell \in J_2 \setminus J_3} I_\ell \lesssim |B|. \quad (113)$$

If $\ell \in J_3$, then by (7),

$$\mathcal{C}(H''_\ell) \gtrsim \frac{I_\ell^3}{|B_\ell|^2 (I_\ell/|\mathcal{P}_\ell|)^{1-1/(k-1)}}. \quad (114)$$

Adding back the $(k-1)|A|$ incidences from curves passing through fewer than k point of B_ℓ , we obtain

$$I_\ell \leq C(\mathcal{C}(H''_\ell)^{\frac{k-1}{2k-1}} |B_\ell|^{\frac{k}{2k-1}} + |A|).$$

Summing over all indices $\ell \in J_3$ and noting that $|J_3| \lesssim \log |B|$, and $|B_\ell| \lesssim 2^{-c\ell} |B|$, we obtain

$$\begin{aligned} \sum_{\ell \in J_3} I_\ell &\lesssim |B|^{k/(2k-1)} \mathcal{C}(U)^{\frac{k-1}{2k-1}} + |A| \\ &\leq |B|^{k/(2k-1)} \mathcal{C}(U_{ij})^{\frac{k-1}{2k-1}} + |A| \log |B|. \end{aligned} \quad (115)$$

Combining (110), (113), and (115), we obtain the lemma. \square

REFERENCES

1. M. Ajtai, V. Chvatal, M. Newborn, E. Szemerédi. Crossing-free subgraphs. *Ann. Discrete Math.* 12:9–12. 1982.
2. B. Aronov, V. Koltun, M. Sharir. Incidences between points and circles in three and higher dimensions. *Discrete Comput. Geom.* 33:185–206. 2005.
3. S. Barone, S. Basu. Refined bounds on the number of connected components of sign conditions on a variety. arXiv:1104.0636v3. 2011.
4. S. Basu, R. Pollack, M. Roy. *Algorithms in real algebraic geometry*. Springer, Berlin. 2006.
5. J. Bochnak, M. Coste, M. Roy. *Real algebraic geometry*. Springer-Verlag, Berlin. 1998.
6. K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, E. Welzl. Combinatorial complexity bounds for arrangements of curves and surfaces. *Discrete Comput. Geom.* 5(1):99–160. 1990.
7. J. Coolidge. *A treatise on algebraic plane curves*. Dover, New York. 1959.
8. H. Edelsbrunner, M. Sharir. A hyperplane incidence problem with applications to counting distances. *The Victor Klee Festschrift, DIMACS Ser. Discret. Math. Theor. Comput. Sci.* 4:253–263. 1991.

9. G. Elekes, C. Tóth. Incidences of not-too-degenerate hyperplanes. *Computational geometry (SCG'05)*. ACM, New York: 16–21. 2005.
10. W. Fulton. *Intersection theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, second edition, 1998.
11. L. Guth, N. Katz. On the Erdős distinct distance problem in the plane. arXiv:1011.4105v1. 2011.
12. R. Harshorne. *Algebraic geometry*. Springer-Verlag, New York. 1983.
13. A. Iosevich, H. Jorati, I. Łaba. Geometric incidence theorems via Fourier analysis. *Trans. Amer. Math. Soc.* 361(12):6595–6611. 2009.
14. H. Kaplan, J. Matoušek, Z. Safernova, M. Sharir. Unit distances in three dimensions. arXiv:1107.1077v1. 2011.
15. H. Kaplan, J. Matoušek, M. Sharir. Simple proofs of classical theorems in discrete geometry via the Guth–Katz polynomial partitioning technique. arXiv:1102.5391v1. 2011.
16. T. Kővari, V. Sós, P. Turán. On a problem of K. Zarankiewicz. *Colloquium Mathematicum*. 3:50–57. 1954.
17. I. Łaba, J. Solymosi. Incidence theorems for pseudoflats. *Discrete Comput. Geom.* 37(2):163–174. 2007.
18. J. Landsberg. Differential-geometric characterizations of complete intersections. *J. Differential Geom.* 44(1):32–73. 1996.
19. F. Leighton. *Complexity Issues in VLSI, Foundations of Computing Series*. MIT Press, Cambridge, MA. 1983.
20. J. Milnor. On the Betti numbers of real varieties. *Proc. AMS.* 15:275–280. 1964.
21. J. Pach, M. Sharir. On the number of incidences between points and curves. *Combin. Probab. Comput.* 7(1):121–127. 1998.
22. J. Pach, M. Sharir. Repeated angles in the plane and related problems. *J. Combin. Theo. Ser. A* 59:12–22. 1992.
23. J. Solymosi, T. Tao. An incidence theorem in higher dimensions. arXiv:1103.2926v2. 2011.
24. J. Solymosi, G. Tardos, On the number of k -rich transformations. *Proceedings of the 23th Annual Symposium on Computational Geometry*, (SoCG 2007), ACM, New York: 227–231. 2007.
25. J. Solymosi, C. Tóth. On distinct distances in homogeneous sets in the Euclidean space. *Discrete Comput. Geom.* 35:537–549. 2005.
26. L. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.* 6(3):353–358. 1997.
27. E. Szemerédi, W. Trotter. Extremal problems in discrete geometry. *Combinatorica*. 3(3):381–392. 1983.
28. R. Thom, Sur l’homologie des variétés algébriques réelles. *Differential and Combinatorial Topology*, (Symposium in Honor of Marston Morse), Ed. S.S. Cairns. Princeton Univ. Press. 255–265. 1965.
29. C. Tóth. The Szemerédi-Trotter theorem in the complex plane. arXiv:math/0305283v3. 2003.
30. H. Whitney. Elementary structure of real algebraic varieties. *Annals of Math.* 66:545–556. 1957.
31. J. Zahl. An improved bound on the number of point-surface incidences in three dimensions. arXiv:1104.4987v3. 2011.

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